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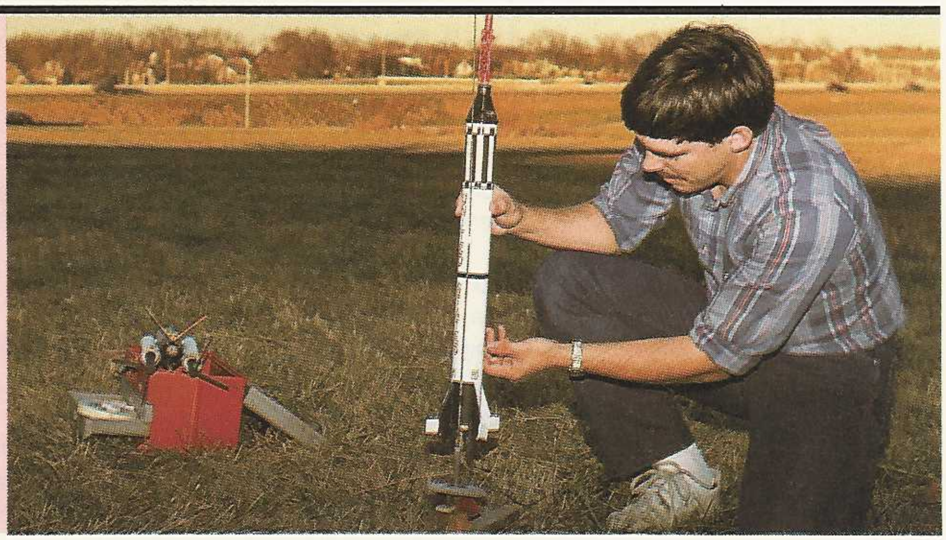
CHAPTER

4 Factoring and Solution of Quadratic Equations by Factoring

The formula $s = vt - 16t^2$ gives the height s in feet that an object will travel in t seconds if it is propelled directly upward at an initial velocity of v feet per second. If an object is thrown upward at 96 feet per second, how long will it take the object to reach a height of 144 feet?



4–1 ■ Common factors

Greatest common factor

To find the solution of certain equations that are not linear, we will need to study a technique called *factoring a polynomial*. Factoring polynomials will also be useful in dealing with algebraic fractions since, as we have seen with arithmetic fractions, we must have the numerator and the denominator of the fractions in a factored form to reduce or to find the least common denominator.

The first type of factoring that we will do involves finding the greatest common factor (GCF) of each term in the polynomial. Recall the statement of the distributive property.

$$a(b + c) = ab + ac$$

$a(b + c)$ is called the *factored form* of $ab + ac$. We now use the distributive property to write $3x + 6$ in the factored form. First we notice that $3x + 6$ can be written as

$$3 \cdot x + 3 \cdot 2 \qquad \text{3 is a common factor in both terms}$$

By applying the distributive property, we have

$$3x + 6 = 3 \cdot x + 3 \cdot 2 = 3(x + 2)$$

$3(x + 2)$ is the factored form of $3x + 6$.

This type of factoring, as its name implies, involves looking for numbers or variables that are **common factors** in *all* of the original terms. In our example, the number 3 was common to all of the original terms, and we were able to factor it out.

When a polynomial is factored, we "factor out" the greatest common factor, GCF. The greatest common factor consists of the following:

Greatest common factor

1. The greatest integer that is a common factor of all the numerical coefficients and
2. the variable factor(s) raised to the least power to which they were raised in any of the terms

← Factoring

Polynomial (terms)	Distributive Property (determine the GCF)	Factored Form (factors)
$3x + 6$	$\mathbf{3} \cdot x + \mathbf{3} \cdot 2$	$3(x + 2)$
$10x^2 + 15y$	$\mathbf{5} \cdot 2x^2 + \mathbf{5} \cdot 3y$	$5(2x^2 + 3y)$
$12a - 42b$	$\mathbf{6} \cdot 2a - \mathbf{6} \cdot 7b$	$6(2a - 7b)$
$18xy + 12xz$	$\mathbf{6x} \cdot 3y + \mathbf{6x} \cdot 2z$	$6x(3y + 2z)$

Multiplying →

In the previous examples, we determined the greatest common factor by inspection. In some problems, this may not be possible, and the following procedure will be necessary. Factor the polynomial $12x^3y + 30x^2y^3$.

Step 1 Factor each term such that it is the product of primes* and variables to powers.

$$12x^3y \qquad\qquad 30x^2y^3$$
$$2^2 \cdot 3 \cdot x^3 \cdot y \qquad\qquad 2 \cdot 3 \cdot 5 \cdot x^2 \cdot y^3$$

Step 2 Write down all the numbers and variables that are common to *every* term.

$$2 \cdot 3 \cdot x \cdot y$$

Note We do not have 5 as part of our greatest common factor since it does not appear as a factor in *all* of the terms.

Step 3 Take the numbers and variables in step 2. Raise them to the *lowest* power to which they were raised in any of the terms.

$$2^1 \cdot 3^1 \cdot x^2 \cdot y^1 = 6x^2y$$

This is the greatest common factor.

Step 4 Find the multinomial factor, the polynomial within the parentheses, by dividing each term of the polynomial being factored by the GCF.

$$\frac{12x^3y}{6x^2y} = 2x \text{ and } \frac{30x^2y^3}{6x^2y} = 5y^2$$

$$6x^2y(2x + 5y^2)$$

Step 5 We can now write the polynomial in its factored form.

$$\begin{aligned} &12x^3y + 30x^2y^3 \\ &= 6x^2y \cdot 2x + 6x^2y \cdot 5y^2 \\ &= 6x^2y(2x + 5y^2) \end{aligned}$$

*Primes and prime factorization are covered in section 1–1.

Completely factored form

In the previous example, $12x^3y + 30x^2y^3$ could also be factored to $3xy(4x^2 + 10xy^2)$ or $12y\left(x^3 + \frac{5}{2}x^2y^2\right)$. This allows room for a given polynomial to be factored in many ways, unless some restrictions are placed on the procedure. We wish to factor each polynomial in a unique manner that will not permit such variations in the results. Thus it is customary to adopt the following criteria for a completely factored polynomial.

A polynomial with integer coefficients will be considered to be in **completely factored form** when it satisfies the following criteria:

Completely factored form

1. The polynomial is written as a product of polynomials with integer coefficients.
2. None of the polynomial factors other than the monomial factor can be further factored.

We see that $6x^2y(2x + 5y^2)$ is the completely factored form of the expression $12x^3y + 30x^2y^3$ since all of the coefficients are integers and, except for the monomial factor $6x^2y$, the remaining factor contains no other factor with integer coefficients.

In general, whenever we factor a monomial out of a polynomial, we factor the monomial so that it has a positive coefficient. Realize that we could also factor out the opposite, or negative, of this common factor. We could have factored out $-6x^2y$ from our example. The completely factored form would have been

$$-6x^2y(-2x - 5y^2)$$

Observe that the only change in our answer when we factor out the opposite of the common factor is that this changes the signs of all terms inside the parentheses.

■ Example 4–1 A

Write in completely factored form.

1. $7a^3 + 14a$

$= 7a(\quad + \quad)$ — Multinomial factor will have as many terms as the original expression; GCF is $7a$

$= 7a(a^2 + 2)$ — Completely factored form

$a^2 = \frac{7a^3}{7a}$, $2 = \frac{14a}{7a}$

If we want to check the answer, we apply the distributive property and perform the multiplication as follows:

$7a(a^2 + 2) = 7a \cdot a^2 + 7a \cdot 2$ — Distributive property

$= 7a^3 + 14a$ — Carry out the multiplication

2. $9x^5 + 6x^3 - 18x^2$

$$\begin{aligned}
&= 3^2x^5 + 2 \cdot 3x^3 - 2 \cdot 3^2x^2 && \text{Factor each term}\\
&= 3x^2(\quad + \quad - \quad) && \text{Determine the GCF}\\
&= 3x^2(3x^3 + 2x - 6) && \text{Completely factored form}
\end{aligned}$$

3. $72a^2b - 84a^3b^4 + 48a^4b^2$

$$\begin{aligned}
&= 2^3 \cdot 3^2 \cdot a^2 \cdot b - 2^2 \cdot 3 \cdot 7 \cdot a^3 \cdot b^4 && \text{Factor each term}\\
&\quad + 2^4 \cdot 3 \cdot a^4 \cdot b^2 &&\\
&= 12a^2b(\quad - \quad + \quad) && \text{Determine the GCF}\\
&= 12a^2b(6 - 7ab^3 + 4a^2b) && \text{Completely factored form}
\end{aligned}$$

4. $3x^3y^2 + 15x^2y^4 + 3xy^2$

$$\begin{aligned}
&= 3 \cdot x^3 \cdot y^2 + 3 \cdot 5 \cdot x^2 \cdot y^4 + 3 \cdot x \cdot y^2 && \text{Factor each term}\\
&= 3xy^2(\quad + \quad + \quad) && \text{Determine the GCF}\\
&= 3xy^2(x^2 + 5xy^2 + 1) && \text{Completely factored form}
\end{aligned}$$

Note In example 4, the last term in the factored form is 1. This situation occurs when a term and the GCF are the same, that is, whenever we are able to factor all of the numbers and variables out of a given term. For example, $\frac{3xy^2}{3xy^2} = 1$. The number of terms inside the parentheses must be equal to the number of terms in the original polynomial.

▶ ***Quick check*** Write $9x + 6y + 3$ and $9x^3y^2 - 18x^2y^3$ in completely factored form. ■

Remember that when an expression is within a grouping symbol, we treat the quantity as just one number. Therefore if we have a quantity common to all of the terms, we can factor it out of the polynomial.

■ Example 4–1 B

Factor completely.

1. $x(a - 2b) + y(a - 2b)$

The quantity $(a - 2b)$ is common to both terms. We then factor the common quantity out of each term and place the remaining factors from each term in the second parentheses.

$$x(a - 2b) + y(a - 2b)$$

$$\underbrace{(a - 2b)}_{\text{Common factor}}\underbrace{(x + y)}_{\text{Remaining factors}}$$

2. $x^2(a + b) + (a + b)$

$$\begin{aligned}
&= (a + b)(\quad + \quad) && \text{Determine the GCF}\\
&= (a + b)(x^2 + 1) && \text{Completely factored form}
\end{aligned}$$

3. $3x^2(2a - b) - 9x(2a - b)$

$$\begin{aligned}
&= 3x^2(2a - b) - 3^2x(2a - b) && \text{Factor each term}\\
&= 3x(2a - b)(\quad - \quad) && \text{Determine the GCF}\\
&= 3x(2a - b)(x - 3) && \text{Completely factored form}
\end{aligned}$$

▶ ***Quick check*** Factor $x(y + 5) - z(y + 5)$ ■

Four-term polynomials

Consider $ax + ay + bx + by$. We observe that this is a *four-term polynomial* and we will *try* to factor it by grouping.

$$ax + ay + bx + by = (ax + ay) + (bx + by)$$

There is a common factor of a in the first two terms and a common factor of b in the last two terms.

$$(ax + ay) + (bx + by) = a(x + y) + b(x + y)$$

The quantity $(x + y)$ is common to both terms. Factoring it out, we have

$$a(x + y) + b(x + y) = (x + y)(a + b)$$

Therefore we have factored the polynomial by grouping.

Factoring a four-term polynomial by grouping

1. Arrange the four terms so that the first two terms have a common factor and the last two terms have a common factor.
2. Determine the GCF of each pair of terms and factor it out.
3. If step 2 produces a common binomial factor in each term, factor it out.
4. If step 2 does not produce a common binomial factor in each term, try grouping the terms of the original polynomial in a different way.
5. If step 4 does not produce a common binomial factor in each term, the polynomial will not factor by this procedure.

Example 4–1 C

Factor completely.

1. $ax + 2ay + bx + 2by$

$= (ax + 2ay) + (bx + 2by)$	Group in pairs
$= a(x + 2y) + b(x + 2y)$	Factor out the GCF
$= (x + 2y)(a + b)$	Factor out the common binomial

2. $3ac + 6ad - 2bc - 4bd$

$= (3ac + 6ad) - (2bc + 4bd)$	Group in pairs
$= 3a(c + 2d) - 2b(c + 2d)$	Factor out the GCF
$= (c + 2d)(3a - 2b)$	Factor out the common binomial

3. $2ax - 2ay + bx - by$

$= (2ax - 2ay) + (bx - by)$	Group in pairs
$= 2a(x - y) + b(x - y)$	Factor out the GCF
$= (x - y)(2a + b)$	Factor out the common binomial

4. $6ax + by + 3ay + 2bx$

$= 6ax + 3ay + 2bx + by$	Rearrange the terms
$= (6ax + 3ay) + (2bx + by)$	Group in pairs
$= 3a(2x + y) + b(2x + y)$	Factor out the GCF
$= (2x + y)(3a + b)$	Factor out the common binomial

Note As in example 4, sometimes the terms must be rearranged so that the pairs will have a common factor.

▶ ***Quick check*** Factor $3ax + 6bx + 2ay + 4by$

It is important to remember to look for the greatest common factor first *when we attempt to determine the completely factored form of any polynomial. If we fail to do this, the answer may not be in a completely factored form or we may not see how to factor the problem by an appropriate procedure.*

Mastery points

Can you

- Determine the greatest common factor?
- Factor a four-term polynomial by grouping?

Exercise 4–1

Write in completely factored form. See example 4–1 A.

Examples $9x + 6y + 3$

Solutions

$= 3 \cdot 3x + 3 \cdot 2y + 3 \cdot 1$ Factor each term
$= 3(\quad + \quad + \quad)$ Determine the GCF
$= 3(3x + 2y + 1)$ Completely factored form

$9x^3y^2 - 18x^2y^3$

$= 9x^2y^2 \cdot x - 9x^2y^2 \cdot 2y$ Factor each term
$= 9x^2y^2(\quad - \quad)$ Determine the GCF
$= \quad (x - 2y)$ Completely factored form

1. $2y + 6$
2. $3a - 12$
3. $4x^2 + 8y$
4. $8y^2 + 10x^2$
5. $3x^2y + 15z$
6. $5r^2 + 10rs - 20s$
7. $7a - 14b + 21c$
8. $8x - 12y + 16z$
9. $15xy - 18z + 3x^2$
10. $18ab - 27a + 3ac$
11. $42xy - 21y^2 + 7$
12. $15a^2 - 27b^2 + 12ab$
13. $8x - 10y + 12z - 18w$
14. $15L^2 - 21W^2 + 36H$
15. $20a^2b - 60ab + 45ab^2$
16. $4x^2 + 8x$
17. $3x^2y + 6xy$
18. $8x^3 + 4x^2$
19. $2R^4 - 6R^2$
20. $3x^2 - 3xy + 3x$
21. $2x^3 - x^2 + x$
22. $24a^2 + 12a - 6a^3$
23. $15ab + 18ab^2 - 3a^2b$
24. $2x^4 - 6x^2 + 8x$
25. $xy^2 + xyz + xy$
26. $3R^2S - 6RS^2 + 12RS$
27. $2L^3 - 18L + 2L^2$
28. $V^2 + V^3 - V^4 + 2V$
29. $5p^2 + 10p + 15p^3$
30. $16x^3y - 3x^2y^2 + 24x^2y^3$

Supply the missing factor.

Example $-3a - 6b = -3(\quad)$

Solution Since $-3a - 6b = (-3) \cdot a + (-3) \cdot 2b$, then $-3a - 6b = -3(a + 2b)$, the missing factor is $(a + 2b)$.

Example $-a^2b^3 + a^2b^2 = -a^2b^2(\quad)$

Solution $-a^2b^3 + a^2b^2 = (-a^2b^2)(b) + (-a^2b^2)(-1)$ Divide each term by $-a^2b^2$ to find the missing factor
$= -a^2b^2(b - 1)$, the missing factor is $(b - 1)$.

31. $-6x - 9 = -3(\quad)$
32. $-5a + 10b = -5(\quad)$
33. $6x - 8z - 12w = 2(\quad)$
34. $-4a^3 - 36ab + 16ab^2 - 24b^3 = -4(\quad)$
35. $-12L + 15W - 6H = -3(\quad)$
36. $-3a + a^3b = -a(\quad)$
37. $-x + x^2 - x^3 = -x(\quad)$
38. $-x + 2xy + xy^2 = -x(\quad)$

39. $-xyz + x^2yz - xy^2z + xyz^2 = -xyz(\quad)$

40. $-4x^2 + 8x - 12x^3 = -4x(\quad)$

41. $-10a^2b^2 + 15ab - 20a^3b^3 = -5ab(\quad)$

42. $-24RS - 16R + 32R^2 = -8R(\quad)$

Write in completely factored form. See example 4–1 B.

Example $x(y + 5) - z(y + 5)$

Solution $= (y + 5)(\quad - \quad)$ Determine the GCF

$= (y + 5)(x - z)$ Completely factored form

43. $x(a + b) + y(a + b)$

44. $3a(x - y) + b(x - y)$

45. $15x(2a + b) + 10y(2a + b)$

46. $21R(L + 2N) - 35S(L + 2N)$

47. $3x(a + 4b) + 6y(a + 4b)$

48. $4RS(2P + q) - 8RT(2P + q)$

49. $8a(b + 6) - (b + 6)$

Write the following in completely factored form. See example 4–1 C.

Example $3ax + 6bx + 2ay + 4by$

Solution $= (3ax + 6bx) + (2ay + 4by)$ Group in pairs

$= 3x(a + 2b) + 2y(a + 2b)$ Factor out the GCF

$= (a + 2b)(3x + 2y)$ Factor out the common binomial

50. $rt + ru + st + su$

51. $ac + ad + bc + bd$

52. $5ax - 3by + 15bx - ay$

53. $6ax - 2by + 3bx - 4ay$

54. $2ax^2 - bx^2 + 6a - 3b$

55. $4ax + 2ay - 2bx - by$

56. $ac + 3ad - 4bc - 12bd$

57. $20x^2 + 5xz - 12xy - 3yz$

58. $a^2x + 3a^2y - 3x - 9y$

59. $4ax + 12bx - 3ay - 9by$

60. $ac + ad - 2bc - 2bd$

61. $2ac + 6bc - ay - 3by$

62. $2ac + bc - 4ay - 2by$

63. $2ac + 3bc + 8ay + 12by$

64. $5ac - 3by + 15bc - ay$

65. $6ax + by + 2ay + 3bx$

66. $2ax - ad + 4bx - 2bd$

67. $3ax - 2bd - 6ad + bx$

68. $6ax + 3bd - 2ad - 9bx$

69. $2a^3 + 15 + 10a^2 + 3a$

70. $3a^3 - 6a^2 + 5a - 10$

71. $8a^3 - 4a^2 + 6a - 3$

Write in completely factored form. See example 4–1 A.

72. The area of the surface of a cylinder is determined by $A = 2\pi rh + 2\pi r^2$. Factor the right member. (π is the Greek letter pi.)

73. The total surface area of a right circular cone is given by $A = \pi rs + \pi r^2$. Factor the right member.

74. The equation for the distance traveled by a rocket fired vertically upward into the air is given by $S = 560t - 16t^2$, where the rocket is S feet from the ground after t seconds. Factor the right member.

75. In engineering, the equation for deflection of a beam is given by

$$Y = \frac{2wx^4}{48EI} - \frac{3\ell wx^3}{48EI} - \frac{\ell^3 wx}{48EI}$$

Factor the right member.

Review exercises

Two numbers are listed. Find two integers such that their product is the first number and their sum is the second number. See section 1–6.

1. 20, 9
2. 12, 7
3. -16, -6
4. -16, 6
5. 16, 10
6. 16, -10
7. 36, 12
8. 11, 12

4–2 ■ Factoring trinomials of the form $x^2 + bx + c$

Determining when a trinomial will factor

In section 3–2, we learned how to multiply two binomials as follows:

Factors Terms

$$(x + 2)(x + 6) = x^2 + 6x + 2x + 12 = x^2 + 8x + 12$$

Multiplying ⟶

In this section, we are going to reverse the procedure and factor the trinomial.

Terms Factors

$$x^2 + 8x + 12 = (x + 2)(x + 6)$$

Factoring ⟶

The following group of trinomials will enable us to see how a trinomial factors.

Product: $12 = 2 \cdot 6$

1. $x^2 + 8x + 12 \quad = \quad (x + 2)(x + 6)$

Sum: $8 = 2 + 6$

Product: $12 = (-2) \cdot (-6)$

2. $x^2 - 8x + 12 \quad = \quad (x - 2)(x - 6)$

Sum: $-8 = (-2) + (-6)$

Product: $-12 = (-2) \cdot 6$

3. $x^2 + 4x - 12 \quad = \quad (x - 2)(x + 6)$

Sum: $4 = (-2) + 6$

Product: $-12 = 2 \cdot (-6)$

4. $x^2 - 4x - 12 \quad = \quad (x + 2)(x - 6)$

Sum: $-4 = 2 + (-6)$

In general,

$$(x + m)(x + n) = x^2 + (m + n)x + m \cdot n$$

The trinomial $x^2 + bx + c$ will factor with integer coefficients only if there are two integers, which we will call m and n, such that $m + n = b$ and $m \cdot n = c$.

Sum Product

$m + n$ $\quad$ $m \cdot n$

$$x^2 + bx + c \quad = \quad (x + m)(x + n)$$

The signs (+ or −) for m and n

1. If c is positive, then m and n have the same sign as b.
2. If c is negative, then m and n have different signs and the one with the greater absolute value has the same sign as b.

Example 4–2 A

Factor completely each trinomial.

1. $a^2 + 11a + 18 \qquad m + n = 11 \quad \text{and} \quad m \cdot n = 18$

Since $b = 11$ and $c = 18$ are both positive, then m and n are both positive.

List the factorizations of 18	Sum of the factors of 18	
$1 \cdot 18$	$1 + 18 = 19$	
$2 \cdot 9$	$2 + 9 = 11$	← Correct sum
$3 \cdot 6$	$3 + 6 = 9$	

The m and n values are 2 and 9. The factorization is

$$a^2 + 11a + 18 = (a + 2)(a + 9)$$

The answer can be checked by performing the indicated multiplication.

$$(a + 2)(a + 9) = a^2 + 9a + 2a + 18 = a^2 + 11a + 18$$

Note The commutative property allows us to write the factors in any order. That is, $(a + 2)(a + 9) = (a + 9)(a + 2)$.

2. $b^2 - 2b - 15 \qquad m + n = -2 \quad \text{and} \quad m \cdot n = -15$

Since $b = -2$ and $c = -15$ are both negative, then m and n have different signs and the one with the greater absolute value is negative.

Factorizations of −15, where the negative sign goes with the factor with the greater absolute value	Sum of the factors of −15	
$1 \cdot (-15)$	$1 + (-15) = -14$	
$3 \cdot (-5)$	$3 + (-5) = -2$	← Correct sum

The m and n values are 3 and −5. The factorization is

$$b^2 - 2b - 15 = (b + 3)(b - 5)$$

3. $5x - 24 + x^2$

It is easier to identify b and c if we write the trinomial in descending powers of the variable, which is called **standard form.**

$$x^2 + 5x - 24 \qquad m + n = 5 \quad \text{and} \quad m \cdot n = -24$$

Since $b = 5$ is positive and $c = -24$ is negative, m and n have different signs and the one with the greater absolute value is positive.

Factorizations of −24, where the positive factor is the one with the greater absolute value	Sum of the factors of −24	
$(-1) \cdot 24$	$(-1) + 24 = 23$	
$(-2) \cdot 12$	$(-2) + 12 = 10$	
$(-3) \cdot 8$	$(-3) + 8 = 5$	← Correct sum
$(-4) \cdot 6$	$(-4) + 6 = 2$	

The m and n values are −3 and 8. The factorization is

$$x^2 + 5x - 24 = (x - 3)(x + 8)$$

4. $c^2 - 9c + 14 \qquad m + n = -9$ and $m \cdot n = 14$

Since $b = -9$ is negative and $c = 14$ is positive, m and n are both negative.

List the factorizations of 14	Sum of the factors of 14
$(-1)(-14)$	$(-1) + (-14) = -15$
$(-2)(-7)$	$(-2) + (-7) = -9 \longleftarrow$ Correct sum

The m and n values are -2 and -7. The factorization is

$$c^2 - 9c + 14 = (c - 2)(c - 7)$$

5. $x^2 + 5x + 12 \qquad m + n = 5$ and $m \cdot n = 12$

Since $b = 5$ and $c = 12$ are both positive, m and n are both positive.

Factorizations of 12	Sum of the factors of 12
$1 \cdot 12$	$1 + 12 = 13$
$2 \cdot 6$	$2 + 6 = 8$
$3 \cdot 4$	$3 + 4 = 7$

No sum equals 5

Since none of the factorizations of 12 add to 5, there is no pair of integers (m and n) and the trinomial will not factor using integer coefficients. We call this a **prime polynomial.**

6. $x^4 - 4x^3 - 21x^2 = x^2(x^2 - 4x - 21)$ Common factor of x^2

To complete the factorization, we see if the trinomial $x^2 - 4x - 21$ will factor. We need to find m and n that add to -4 and multiply to -21. The values are 3 and -7. The completely factored form is

$$x^4 - 4x^3 - 21x^2 = x^2(x + 3)(x - 7)$$

Note A common error when the polynomial has a common factor is to factor it out but to forget to include it as one of the factors in the completely factored form.

7. $x^2y^2 + 9xy + 20$

Rewriting the polynomial as $(xy)^2 + 9xy + 20$, we want to find values for m and n that add to 9 and multiply to 20. The numbers are 4 and 5. The factorization is

$$x^2y^2 + 9xy + 20 = (xy + 4)(xy + 5)$$

8. $x^2 - 5ax + 6a^2 \qquad m + n = -5a$ and $m \cdot n = 6a^2$

We need to find m and n that add to $-5a$ and multiply to $6a^2$. The values are $-2a$ and $-3a$. The factorization is

$$x^2 - 5ax + 6a^2 = (x - 2a)(x - 3a)$$

▶ ***Quick check*** Factor $z^2 + 8z - 20$ ■

Factoring a trinomial of the form $x^2 + bx + c$

1. Factor out the GCF. If there is a common factor, make sure to include it as part of the final factorization.
2. Determine if the trinomial is factorable by finding m and n such that $m + n = b$ and $m \cdot n = c$. If m and n do not exist, we conclude that the trinomial will not factor.
3. Using the m and n values from step 2, write the trinomial in factored form.

Mastery points

Can you

- Determine two integers whose product is one number and whose sum is another number?
- Recognize when the trinomial $x^2 + bx + c$ will factor and when it will not?
- Factor trinomials of the form $x^2 + bx + c$?
- Always remember to look for the greatest common factor before applying any of the factoring rules?

Exercise 4–2

Factor completely each trinomial. See example 4–2 A.

Example $z^2 + 8z - 20$

Solution Since $b = 8$ is positive and $c = -20$ is negative, m and n have different signs and the one with the greater absolute value is positive.

Factorizations of -20, where the positive factor is the one with the greater absolute value	Sum of the factors of -20
$(-1) \cdot 20$	$(-1) + 20 = 19$
$(-2) \cdot 10$	$(-2) + 10 = 8 \longleftarrow$ Correct sum
$(-4) \cdot 5$	$(-4) + 5 = 1$

The m and n values are -2 and 10. The factorization is

$z^2 + 8z - 20 = (z - 2)(z + 10)$

1. $a^2 + 9a + 18$
2. $c^2 + 9c + 20$
3. $x^2 + 11x - 12$
4. $x^2 + 13x + 12$
5. $y^2 + 13y - 30$
6. $a^2 + 9a + 14$
7. $x^2 - 14x + 24$
8. $b^2 - 10b + 21$
9. $a^2 + 5a - 24$
10. $y^2 + 9y - 36$
11. $x^2 + 8x + 12$
12. $c^2 + 8c + 15$
13. $a^2 - 2a - 24$
14. $z^2 - 5z - 36$
15. $2x^2 + 6x - 20$
16. $2a^2 + 26a + 24$
17. $3x^2 - 18x - 48$
18. $a^2 - 9a + 4$
19. $x^2 + 5x + 7$
20. $x^2 - 4x + 6$
21. $y^2 + 17y + 30$
22. $b^2 + 13b + 40$
23. $4x^2 - 4x - 24$
24. $5y^2 + 5y - 30$
25. $5a^2 - 15a - 50$
26. $x^2y^2 - 4xy - 21$
27. $x^2y^2 - 3xy - 18$
28. $x^2y^2 - xy - 30$
29. $x^2y^2 + 13xy + 12$
30. $4a^2b^2 - 32ab + 28$
31. $3x^2y^2 - 3xy - 36$
32. $3x^2y^2 + 21xy + 36$
33. $x^2 + 3xy + 2y^2$
34. $a^2 - ab - 2b^2$
35. $a^2 - 2ab - 3b^2$
36. $a^2 - 7ab + 10b^2$
37. $a^2 - ab - 6b^2$
38. $x^2 + 2xy - 8y^2$
39. $x^2 - 2xy - 15y^2$
40. $a^2 + 7ab + 12b^2$

Review exercises

Factor completely. See section 4–1.

1. $ax^2 + bx^2 + cx^2$
2. $3x^3 + 12x^2 - 6x$
3. $3x(2x + 1) + 5(2x + 1)$
4. $2x(3x - 2) + 3(3x - 2)$
5. $4x(5x + 1) + (5x + 1)$
6. $6x(2x + 3) - (2x + 3)$
7. $x(3x - 5) - 2(3x - 5)$
8. $7x(x - 9) - 3(x - 9)$

4–3 ■ Factoring trinomials of the form $ax^2 + bx + c$

How to factor trinomials

In this section, we are going to factor trinomials of the form $ax^2 + bx + c$. This is called the **standard form** of a trinomial, where we have a single variable and the terms of the polynomial are arranged in descending powers of that variable. The a, b, and c in our standard form represent integer constants. For example,

$$2x^2 + 9x + 9$$

is a trinomial in standard form, where $a = 2$, $b = 9$, and $c = 9$.

Consider the product

$$(2x + 3)(x + 3)$$

By multiplying these two quantities together, we get a trinomial.

$$\begin{aligned}(2x + 3)(x + 3) &= 2x^2 + 6x + 3x + 9\\ &= 2x^2 + 9x + 9\end{aligned}$$

To completely factor the trinomial $2x^2 + 9x + 9$ entails reversing this procedure to get

$$(2x + 3)(x + 3)$$

The trinomial will factor with integer coefficients if we can find a pair of integers (m and n) whose sum is equal to b, and whose product is equal to $a \cdot c$. In the trinomial $2x^2 + 9x + 9$, b is equal to 9, and $a \cdot c$ is $2 \cdot 9 = 18$. Therefore we want $m + n = 9$ and $m \cdot n = 18$. The values for m and n are 3 and 6.

If we observe the multiplication process in our example, we see that m and n appear as the coefficients of the middle terms that are to be combined for our final answer.

$$\begin{aligned}(2x + 3)(x + 3) &= 2x^2 + 6x + 3x + 9\\ &= 2x^2 + 9x + 9\end{aligned}$$

This is precisely what we do with the m and n values. We replace the coefficient of the middle term in the trinomial with these values. In our example, m and n are 6 and 3 and we replace the 9 with them.

$$2x^2 + 9x + 9 = 2x^2 + \overbrace{6x + 3x}^{9x} + 9$$

Our next step is to group the first two terms and the last two terms.

$$(2x^2 + 6x) + (3x + 9)$$

Now we factor out what is common in each pair. We see that the first two terms contain the common factor $2x$ and the last two terms contain the common factor 3.

$$2x(x + 3) + 3(x + 3)$$

When we reach this point, what is inside the parentheses in each term will be the same. Since the quantity $(x + 3)$ is common to both terms, we can factor it out.

$$2x\underbrace{(x + 3)} + 3\underbrace{(x + 3)}$$

Common to both terms

Having factored out what is common, what is left in each term is placed in the second parentheses.

$$2x(x + 3) + 3(x + 3)$$

$$\underset{\text{Common factor}}{(x + 3)}\underset{\text{Remaining factors}}{(2x + 3)}$$

The trinomial is factored.

A summary of the steps follows:

Factoring a trinomial of the form $ax^2 + bx + c$

Step 1 Determine if the trinomial $ax^2 + bx + c$ is factorable by finding m and n such that $m \cdot n = a \cdot c$ and $m + n = b$. If m and n do not exist, we conclude that the trinomial will not factor.

Step 2 Replace the middle term, bx, by the sum of mx and nx.

Step 3 Place parentheses around the first and second terms and around the third and fourth terms. Factor out what is common to each pair.

Step 4 Factor out the common quantity of each term and place the remaining factors from each term in the second parentheses.

We determine the signs (+ or −) for m and n in a fashion similar to that of section 4–2.

The signs (+ or −) for m and n

1. If $a \cdot c$ is positive, then m and n have the same sign as b.
2. If $a \cdot c$ is negative, then m and n have different signs and the one with the greater absolute value has the same sign as b.

■ Example 4–3 A

Factor completely the following trinomials. If a trinomial will not factor, so state.

1. $6x^2 + 13x + 6$

Step 1 $m \cdot n = 6 \cdot 6 = 36$ and $m + n = 13$

We determine by inspection that m and n are 9 and 4.

Step 2 $= 6x^2 \overbrace{+ 9x + 4x}^{13x} + 6$ — Replace bx with mx and nx

Step 3 $= (6x^2 + 9x) + (4x + 6)$ — Group the first two terms and the last two terms

$= 3x(2x + 3) + 2(2x + 3)$ — Factor out what is common to each pair

Step 4 $= (2x + 3)(3x + 2)$ — Factor out the common quantity

Note The order in which we place m and n into the problem will not change the answer.

(Alternate)

Step 2 $= 6x^2 \overbrace{+ 4x + 9x}^{13x} + 6$ — Replace bx with nx and mx

Step 3 $= (6x^2 + 4x) + (9x + 6)$ — Group the first two terms and the last two terms

$= 2x(3x + 2) + 3(3x + 2)$ — Factor out what is common to each pair

Step 4 $= (3x + 2)(2x + 3)$ — Factor out the common quantity

We see that the outcome in step 4 is the same regardless of the order of m and n in the problem.

Note The order in which the two factors are written in the answer does not matter. That is, $(2x + 3)(3x + 2) = (3x + 2)(2x + 3)$

2. $3x^2 + 5x + 2$

Step 1 $m \cdot n = 3 \cdot 2 = 6$ and $m + n = 5$

m and n are 2 and 3.

Step 2 $= 3x^2 \overbrace{+ 2x + 3x}^{5x} + 2$ — Replace bx with mx and nx

Step 3 $= (3x^2 + 2x) + (3x + 2)$ — Group the first two terms and the last two terms

$= x(3x + 2) + 1(3x + 2)$ — Factor out what is common to each pair

We observe in the last two terms that the greatest common factor is only 1 or -1. We factor out 1 so that we have the same quantity inside the parentheses.

Step 4 $= (3x + 2)(x + 1)$ — Factor out the common quantity

3. $4x^2 - 11x + 6$

Step 1 $m \cdot n = 4 \cdot 6 = 24$ and $m + n = -11$

m and n are -3 and -8.

Step 2 $= 4x^2 \overbrace{- 3x - 8x}^{-11x} + 6$ — Replace bx with mx and nx

Step 3 $= (4x^2 - 3x) + (-8x + 6)$ Group the first two terms and the last two terms

$= x(4x - 3) - 2(4x - 3)$ Factor out what is common to each pair

We have 2 or -2 as the greatest common factor in the last two terms. We factor out -2 so that we will have the same quantity inside the parentheses.

Step 4 $= (4x - 3)(x - 2)$ Factor out the common quantity

Note If the third term in step 2 is preceded by a minus sign, we will usually factor out the negative factor.

4. $12x^2 - 4x - 5$

Step 1 $m \cdot n = 12(-5) = -60$ and $m + n = -4$

m and n are 6 and -10.

Step 2 $= 12x^2 + \overbrace{6x - 10x}^{-4x} - 5$ Replace bx with mx and nx

Step 3 $= (12x^2 + 6x) + (-10x - 5)$ Group the first two terms and the last two terms

$= 6x(2x + 1) - 5(2x + 1)$ Factor out what is common to each pair

Step 4 $= (2x + 1)(6x - 5)$ Factor out the common quantity

5. $6x^2 - 9x - 4$

$m \cdot n = 6 \cdot (-4) = -24$ and $m + n = -9$

Our m and n values are not obvious by inspection.

Note If you cannot determine the m and n values by inspection, then you should use the following systematic procedure to list all the possible factorizations of $a \cdot c$. This way you will either find m and n or verify that the trinomial will not factor using integers.

1. Take the natural numbers $1,2,3,4, \cdot \cdot \cdot$ and divide them into the $a \cdot c$ product. Those that divide into the product evenly we write as a factorization using the correct m and n signs.

Factorization of -24, where the negative sign goes with the factor with the greater absolute value

$$\left.\begin{matrix} 1 \cdot (-24) \\ 2 \cdot (-12) \\ 3 \cdot (-8) \\ 4 \cdot (-6) \end{matrix}\right\}$$
$$\left.\begin{matrix} (-6) \cdot 4 \\ (-8) \cdot 3 \\ (-12) \cdot 2 \\ (-24) \cdot 1 \end{matrix}\right\}$$

We note that the top four factorizations are the same as the bottom four. Therefore we need only perform this procedure until the factors repeat

$4 \cdot (-6)$

$(-6) \cdot 4$

Factors repeat

2. Find the sum of the factorizations of $a \cdot c$. If there is a sum equal to b, the trinomial will factor. If there is no sum equal to b, then the trinomial will not factor with integer coefficients.

Factorizations of -24	Sum of the factors of -24	
$1 \cdot (-24)$	$1 + (-24) = -23$	
$2 \cdot (-12)$	$2 + (-12) = -10$	
$3 \cdot (-8)$	$3 + (-8) = -5$	← Passed -9
$4 \cdot (-6)$	$4 + (-6) = -2$	
	No sum equals -9	

Since none of the factorizations of -24 add to -9, there is no pair of integers (m and n) and the trinomial will not factor.

Note Regardless of the signs of m and n, the column of values of the sum of the factors will either be increasing or decreasing. Therefore, once the desired value has been passed, the process can be stopped and the trinomial will not factor.

6. $24x^2 - 39x - 18$

Before we attempt to apply any factoring rule, recall that we must always factor out what is common to each term. Therefore we have

$24x^2 - 39x - 18 = 3(8x^2 - 13x - 6)$ Common factor of 3

Now we are ready to factor the trinomial $8x^2 - 13x - 6$.

Step 1 $m \cdot n = 8(-6) = -48$ and $m + n = -13$
m and n are 3 and -16.

$$\begin{aligned}
\textbf{\textit{Step 2}} &\quad = 3(8x^2 \overbrace{+ 3x - 16x}^{-13x} - 6) && \text{Replace } bx \text{ with } mx \text{ and } nx\\
\textbf{\textit{Step 3}} &\quad = 3[(8x^2 + 3x) + (-16x - 6)] && \text{Group the first two terms and the last two terms}\\
&\quad = 3[x(8x + 3) - 2(8x + 3)] && \text{Factor out what is common to each pair}\\
\textbf{\textit{Step 4}} &\quad = 3(8x + 3)(\quad - 2) && \text{Factor out the common quantity}
\end{aligned}$$

Note In example 6, we factored out 3 that was common to all the original terms. A common error is to forget to include it as one of the factors in the answer.

▶ ***Quick check*** Factor $6x^2 + 23x + 15$ and $12x^2 + 12x - 9$ ■

Factoring by inspection—an alternative approach

In the beginning of this section, we studied a systematic procedure for determining if a trinomial will factor and how to factor it. In many instances, we can determine how the trinomial will factor by inspecting the problem rather than by applying this procedure.

Factoring by inspection is accomplished as follows: Factor $7x + 2x^2 + 3$.

Step 1 Write the trinomial in standard form.

$2x^2 + 7x + 3$ Arrange terms in descending powers of x

Step 2 Determine the possible combinations of first-degree factors of the first term.

$(2x \quad)(x \quad)$ The only factorization of $2x^2$ is $2x \cdot x$

Step 3 Combine with the factors of step 2 all the possible factors of the third term.

$(2x \quad 3)(x \quad 1)$ The only factorization of 3 is $3 \cdot 1$
$(2x \quad 1)(x \quad 3)$

Step 4 Determine the possible symbol (+ or −) between the terms in each binomial.

$(2x + 3)(x + 1)$
$(2x + 1)(x + 3)$

The rules of real numbers given in chapter 1 provide the answer to step 4.

1. If the third term is preceded by a + sign and the middle term is preceded by a + sign, then the symbols will be

   $(\quad + \quad)(\quad + \quad)$

2. If the third term is preceded by a + sign and the middle term is preceded by a − sign, then the symbols will be

   $(\quad - \quad)(\quad - \quad)$

3. If the third term is preceded by a − sign, then the symbols will be

   $(\quad + \quad)(\quad - \quad)$

   or

   $(\quad - \quad)(\quad + \quad)$

Note It is assumed that the first term is preceded by a + sign or no sign. If it is preceded by a − sign, these rules could still be used if (-1) is first factored out of all the terms.

Step 5 Determine which factors, if any, yield the correct middle term.

$(2x + 3)(x + 1)$

$+3x$ $\qquad (+3x) + (+2x) = +5x$

$+2x$

$(2x + 1)(x + 3)$

$+x$ $\qquad (+x) + (+6x) = +7x$ Correct middle term

$+6x$

The second set of factors gives us the correct middle term. Therefore

$(2x + 1)(x + 3)$ is the factorization of $2x^2 + 7x + 3$.

Example 4–3 B Factor completely the following trinomials by inspection.

1. $6x^2 + 17x + 5$

Step 1 $6x^2 + 17x + 5$ — Standard form

Step 2 $(6x \quad)(x \quad)$ — $6x^2 = 3x \cdot 2x$ or $6x \cdot x$

$(3x \quad)(2x \quad)$

Step 3 $(6x \quad 5)(x \quad 1)$

$(6x \quad 1)(x \quad 5)$ — The only factorization of 5 is $5 \cdot 1$

$(3x \quad 5)(2x \quad 1)$

$(3x \quad 1)(2x \quad 5)$

Step 4 $(6x + 5)(x + 1)$

$(6x + 1)(x + 5)$ — Using the rules of signed numbers, determine the possible signs between the terms

$(3x + 5)(2x + 1)$

$(3x + 1)(2x + 5)$

Step 5 $(6x + 5)(x + 1)$

$+5x$ — $(+5x) + (+6x) = +11x$

$+6x$

$(6x + 1)(x + 5)$

$+x$ — $(+x) + (+30x) = +31x$

$+30x$

$(3x + 5)(2x + 1)$

$+10x$ — $(+10x) + (+3x) = +13x$

$+3x$

$(3x + 1)(2x + 5)$

$+2x$ — $(+2x) + (+15x) = +17x$ Correct middle term

$+15x$

The last set of factors gives us the correct middle term. Hence $(3x + 1)(2x + 5)$ is the factorization of $6x^2 + 17x + 5$.

2. $4x^2 - 5x + 1$

Step 1 $4x^2 - 5x + 1$ — Standard form

Step 2 $(4x \quad)(x \quad)$ — $4x^2 = 2x \cdot 2x$ or $4x \cdot x$

$(2x \quad)(2x \quad)$

Step 3 $(4x \quad 1)(x \quad 1)$ — The only factorization of 1 is $1 \cdot 1$

$(2x \quad 1)(2x \quad 1)$

Step 4 $(4x - 1)(x - 1)$ — Determine the possible signs between the terms

$(2x - 1)(2x - 1)$

Step 5 $(4x - 1)(x - 1)$

$-x$

$-4x$

$(-x) + (-4x) = -5x$ Correct middle term

$(2x - 1)(2x - 1)$

$-2x$

$-2x$

$(-2x) + (-2x) = -4x$

The first set of factors gives us the correct factorization.

$4x^2 - 5x + 1 = (4x - 1)(x - 1)$

3. $13x - 5 + 6x^2$

Step 1 $6x^2 + 13x - 5$ Write in standard form

Step 2 $(6x \quad)(x \quad)$ $6x^2 = 6x \cdot x$ or $2x \cdot 3x$

$(2x \quad)(3x \quad)$

Step 3 $(6x \quad 5)(x \quad 1)$

$(6x \quad 1)(x \quad 5)$ The only factorization of 5 is $5 \cdot 1$

$(2x \quad 5)(3x \quad 1)$

$(2x \quad 1)(3x \quad 5)$

Step 4 $(6x + 5)(x - 1)$ or $(6x - 5)(x + 1)$ Determine the possible signs between the terms

$(6x + 1)(x - 5)$ or $(6x - 1)(x + 5)$

$(2x + 5)(3x - 1)$ or $(2x - 5)(3x + 1)$

$(2x + 1)(3x - 5)$ or $(2x - 1)(3x + 5)$

Step 5 $(6x + 5)(x - 1)$ or $(6x - 5)(x + 1)$

$+5x$ $\qquad -5x$

$-6x$ $\qquad +6x$

$(+5x) + (-6x) = -x$ or $(-5x) + (+6x) = +x$

$(6x + 1)(x - 5)$ or $(6x - 1)(x + 5)$

$+x$ $\qquad -x$

$-30x$ $\qquad +30x$

$(+x) + (-30x) = -29x$ or $(-x) + (+30x) = +29x$

$(2x + 5)(3x - 1)$ or $(2x - 5)(3x + 1)$

$+15x$ $\qquad -15x$

$-2x$ $\qquad +2x$

$(+15x) + (-2x) = +13x$ or $(-15x) + (+2x) = -13x$

Correct middle term

$$(2x+1)(3x-5) \text{ or } (2x-1)(3x+5)$$

$$+3x \qquad\qquad -3x$$

$$-10x \qquad\qquad +10x$$

$$(+3x)+(-10x)=-7x \text{ or } (-3x)+(+10x)=+7x$$

The factorization of $6x^2 + 13x - 5$ is $(2x + 5)(3x - 1)$.

Mastery points

Can you

- Determine two integers whose product is one number and whose sum is another number?
- Recognize when the trinomial $ax^2 + bx + c$ will factor and when it will not?
- Factor trinomials of the form $ax^2 + bx + c$?
- Always remember to look for the greatest common factor before applying any of the factoring rules?

Exercise 4–3

Factor completely each trinomial. If a trinomial will not factor, so state. See examples 4–3 A and B.

Example $6x^2 + 23x + 15$

Solution $m \cdot n = 6 \cdot 15 = 90$ — Determine m and n

$m + n = 23$

m and n are 5 and 18.

$= 6x^2 + \overbrace{5x + 18x}^{23x} + 15$ — Replace bx with mx and nx

$= (6x^2 + 5x) + (18x + 15)$ — Group first two terms and last two terms

$= x(6x + 5) + 3(6x + 5)$ — Factor out what is common to each pair

$= (6x + 5)(x + 3)$ — Factor out the common quantity

Example $12x^2 + 12x - 9$

Solution $= 3(4x^2 + 4x - 3)$ — Common factor of 3

$m \cdot n = 4(-3) = -12$

$m + n = 4$

m and n are -2 and 6. — Determine m and n

$= 3[4x^2 \overbrace{- 2x + 6x}^{4x} - 3]$ — Replace bx with mx and nx

$= 3[2x(2x - 1) + 3(2x - 1)]$ — Group first two terms and last two terms, factor out what is common to each pair

$= 3(2x - 1)(2x + 3)$ — Factor out the common quantity

1. $2x^2 + x - 6$
2. $3x^2 + 7x - 6$
3. $2x^2 + 3x + 1$
4. $4x^2 - 5x + 1$
5. $2R^2 - 7R + 6$
6. $R^2 - 4R + 6$
7. $5x^2 - 7x - 6$
8. $2x^2 - x - 1$

9. $9x^2 - 6x + 1$
10. $8x^2 - 17x + 2$
11. $5x^2 + 4x + 6$
12. $2x^2 - 11x + 12$
13. $6x^2 + 13x + 6$
14. $2r^2 + 13r + 18$
15. $4x^2 + 20x + 21$
16. $7R^2 + 20R - 3$
17. $4x^2 - 2x + 5$
18. $4x^2 - 4x - 3$
19. $9y^2 - 21y - 8$
20. $6x^2 - 23x - 4$
21. $10x^2 + 7x - 6$
22. $10x^2 + 9x + 2$
23. $2x^2 - 9x + 10$
24. $7x^2 - 3x + 6$
25. $4x^2 + 14x + 12$
26. $5R^2 - 9R - 2$
27. $4x^2 + 10x + 6$
28. $6x^2 - 17x + 12$
29. $6x^2 + 7x - 3$
30. $3x^2 + 12x + 12$
31. $2x^2 + 6x - 20$
32. $3a^2 + 8a - 4$
33. $6x^2 + 5x - 6$
34. $3x^2 - 19x + 20$
35. $4x^2 + 12x + 9$
36. $9z^2 - 30z + 25$
37. $7x^2 - 36x + 5$
38. $3x^2 + 2x + 4$
39. $15P^2 + 2P - 1$
40. $12x^2 + 13x - 4$
41. $2x^3 - 6x^2 - 20x$
42. $4x^2 + 10x + 4$
43. $2a^3 + 15a^2 + 7a$
44. $9x^2 + 27x + 8$
45. $8x^2 - 14x - 15$
46. $8x^2 - 18x + 9$
47. When a stone is thrown vertically into the air, the height S of the stone at any instant in time t is given by $S = -16t^2 + 32t - 16$. Factor the right member.

Review exercises

Use the special product rules to carry out the indicated multiplication. See section 3–2.

1. $(x - y)(x + y)$
2. $(3a - 2b)(3a + 2b)$
3. $(x - y)^2$
4. $(5a + 4b)(5a - 4b)$
5. $(2a + b)^2$
6. $(4x - y)^2$
7. $(x^2 + 1)(x^2 - 1)$
8. $(a^2 - 4)(a^2 + 4)$

4–4 Factoring the difference of two squares and perfect square trinomials

In section 3–2, we saw that the product of $(a + b)(a - b)$ was $a^2 - b^2$. We refer to the indicated product $(a + b)(a - b)$ as the *product* of the *sum* and *difference* of the same two terms. Notice that in one factor we *add* the terms and in the other we find the *difference* between these same terms. The product will *always* be the *difference of the squares* of the two terms. To factor the **difference of two squares,** we reverse the formula from section 3–2.

Factoring the difference of two squares

$$a^2 - b^2 = (a + b)(a - b)$$

Concept

(first term)^2 − (second term)^2 factors into
(first term + second term)(first term − second term)

To use this factoring technique, we must be able to recognize **perfect squares.**

Example 4–4 A

Write the following as a quantity squared, if possible.

1. $16 = 4 \cdot 4$ — 16 is a perfect square
   $= (4)^2$ — 16 can be written as 4 squared
2. $x^2 = x \cdot x$ — x is written as a factor twice
   $= (x)^2$ — Writing x^2 as $(x)^2$ shows this is a perfect square
3. $25a^2 = 5a \cdot 5a$ — 25 is $5 \cdot 5$ and a^2 is $a \cdot a$
   $= (5a)^2$ — It is now rewritten as a square
4. $9y^4 = 3y^2 \cdot 3y^2$ — 9 is $3 \cdot 3$ and y^4 could be written as $y^2 \cdot y^2$
   $= (3y^2)^2$ — It is now rewritten as a square

▶ ***Quick check*** Write 64 and $9x^4$ each as a quantity squared.

This is the procedure we use for factoring the difference of two squares.

Factoring the difference of two squares

Step 1 Identify that we have a perfect square *minus* another perfect square.

Step 2 Rewrite the problem as a first term squared minus a second term squared.

(first term)^2 − (second term)^2

Step 3 Factor the problem into the first term plus the second term times the first term minus the second term.

(first term + second term)(first term − second term)

Example 4–4 B

Write the following in completely factored form.

	Step 1 Identify		*Step 2* Rewrite		*Step 3* Factor
	$a^2 - b^2$	$=$	$(a)^2 - (b)^2$	$=$	$(a + b)(a - b)$
1.	$x^2 - 9$	$=$	$(x)^2 - (3)^2$	$=$	$(x + 3)(x - 3)$
2.	$4a^2 - b^2$	$=$	$(2a)^2 - (b)^2$	$=$	$(2a + b)(2a - b)$
3.	$4p^2 - 25v^2$	$=$	$(2p)^2 - (5v)^2$	$=$	$(2p + 5v)(2p - 5v)$
4.	$r^4 - 49$	$=$	$(r^2)^2 - (7)^2$	$=$	$(r^2 + 7)(r^2 - 7)$

▶ ***Quick check*** Write $t^2 - 64$ and $4a^2 - b^2c^2$ in completely factored form.

Our first step in any factoring problem is to look for any common factors. Often an expression that does not appear to be factorable becomes so by taking out the greatest common factor. When we have applied a factoring rule to a problem, *we must inspect all parts of our answer to make sure that nothing will factor further.*

Example 4–4 C

Write the following in completely factored form.

1. $2x^2 - 18y^2 = 2(x^2 - 9y^2)$ — Factor out what is common, 2
   $= 2[(x)^2 - (3y)^2]$ — Identify and rewrite as squares
   $= 2(x + 3y)(x - 3y)$ — Factor and inspect the factors

2. $3w^2 - 48 = 3(w^2 - 16)$ Common factor of 3
$= 3[(w)^2 - (4)^2]$ Identify and rewrite
$= 3(w + 4)(w - 4)$ Factor and inspect the factors

3. $5a^4 - 45a^2b^2 = 5a^2(a^2 - 9b^2)$ Common factor of $5a^2$
$= 5a^2[(a)^2 - (3b)^2]$ Identify and rewrite
$= 5a^2(a + 3b)(a - 3b)$ Factor and inspect the factors

4. $a^4 - 16 = (a^2)^2 - (4)^2$ Identify and rewrite
$= (a^2 + 4)(a^2 - 4)$ Factor and inspect the factors
$= (a^2 + 4)[(a)^2 - (2)^2]$ Identify and rewrite
$= (a^2 + 4)(a + 2)(a - 2)$ Factor and inspect

Note In example 4, $a^2 + 4$ is called the *sum of two squares*. This will *not* factor using integers.

5. $2x^4 - 162 = 2(x^4 - 81)$ Common factor of 2
$= 2[(x^2)^2 - (9)^2]$ Identify and rewrite
$= 2(x^2 + 9)(x^2 - 9)$ Factor and inspect
$= 2(x^2 + 9)[(x)^2 - (3)^2]$ Identify and rewrite
$= 2(x^2 + 9)(x + 3)(x - 3)$ Factor and inspect

Note A common error in examples 1, 2, 3, and 5 is to factor out something that is common but forget to include it as a factor in the final answer.

Perfect square trinomials

In section 3–2, two of the special products that we studied were the squares of a binomial. We will now restate those special products.

$$a^2 + 2ab + b^2 = (a + b)^2$$

and

$$a^2 - 2ab + b^2 = (a - b)^2$$

The right members of the equations are called the squares of binomials, and the left members are called perfect square trinomials. Perfect square trinomials can always be factored by our factoring procedure. However if we observe that the first and last terms of a trinomial are perfect squares, we should see if the trinomial will factor as the square of a binomial. To factor a trinomial as a perfect square trinomial, the following three conditions need to be met.

Necessary conditions for a perfect square trinomial

1. The first term must have a positive coefficient and be a perfect square, a^2.
2. The last term must have a positive coefficient and be a perfect square, b^2.
3. The middle term must be twice the product of the bases of the first and last terms, $2ab$ or $-2ab$.

We observe that

$$9x^2 + 12x + 4$$
$$= \underbrace{(3x)^2}_{\text{Condition 1}} + \underbrace{2(3x)(2)}_{\text{Condition 3}} + \underbrace{(2)^2}_{\text{Condition 2}}$$

Therefore it is a perfect square trinomial and factors into

$$(3x + 2)^2$$

Example 4–4 D

The following examples show the factoring of some other perfect square trinomials.

	Condition 1		Condition 3		Condition 2		Square of a binomial
1. $4x^2 + 20x + 25 =$	$(2x)^2$	$+$	$2(2x)(5)$	$+$	$(5)^2$	$=$	$(2x + 5)^2$
2. $9x^2 - 6x + 1 =$	$(3x)^2$	$-$	$2(3x)(1)$	$+$	$(1)^2$	$=$	$(3x - 1)^2$
3. $16x^2 + 24x + 9 =$	$(4x)^2$	$+$	$2(4x)(3)$	$+$	$(3)^2$	$=$	$(4x + 3)^2$
4. $9y^2 - 30y + 25 =$	$(3y)^2$	$-$	$2(3y)(5)$	$+$	$(5)^2$	$=$	$(3y - 5)^2$

Mastery points

Can you

- Identify and rewrite a perfect square?
- Factor the difference of two squares?
- Remember that the sum of two squares will not factor?
- Factor out any common factors before applying other factoring rules?
- Inspect all factors to make sure the problem is completely factored?
- Factor perfect square trinomials?

Exercise 4–4

Write the following as a quantity squared, if possible. See example 4–4 A.

Examples 64

Solutions $= 8 \cdot 8$ Identify
$= (8)^2$ Rewrite

$9x^4$

$= 3x^2 \cdot 3x^2$ Identify
$= (3x^2)^2$ Rewrite

1. 36 **2.** 25 **3.** c^2 **4.** e^2
5. $16x^2$ **6.** $49b^2$ **7.** $4z^4$ **8.** $25b^2$

Write in completely factored form. See examples 4–4 B, C, and D.

Examples $t^2 - 64$ Identify

Solutions $= (t)^2 - (8)^2$ Rewrite
$= (t + 8)(t - 8)$ Factor

$4a^2 - b^2c^2$ Identify

$= (2a)^2 - (bc)^2$ Rewrite
$= (2a + bc)(2a - bc)$ Factor

9. $x^2 - 1$
10. $x^2 - 25$
11. $a^2 - 4$
12. $r^2 - s^2$
13. $9 - E^2$
14. $49 - R^2$
15. $1 - k^2$
16. $4y^2 - 9$
$\boxed{17.}$ $9b^2 - 16$
18. $x^2 - 16z^2$
19. $b^2 - 36c^2$
20. $16x^2 - y^2$
21. $4a^2 - 25b^2$
22. $16a^2 - b^2$
23. $25p^2 - 81$
24. $r^2 - 4s^2$
$\boxed{25.}$ $8x^2 - 32y^2$
26. $3a^2 - 27b^2$
27. $5r^2 - 125s^2$
28. $20 - 5b^2$
29. $50 - 2x^2$
30. $x^2y^2 - 4z^2$
31. $r^2s^2 - 25t^2$
32. $a^4 - 25$
33. $x^4 - 9$
$\boxed{34.}$ $x^4 - 1$
35. $r^4 - 81$
36. $16t^4 - 1$
37. $49x^2 - 64y^4$
38. $125p^2 - 20v^2$
39. $98x^2y^2 - 50p^2c^2$
40. $a^2 + 10a + 25$
41. $c^2 - 14c + 49$
42. $b^2 + 8b + 16$
43. $a^2 + 6a + 9$
44. $x^2 - 12x + 36$
45. $y^2 - 6y + 9$
46. $a^2 + 6ab + 9b^2$
47. $4a^2 - 12ab + 9b^2$
48. $x^2 - 16xy + 64y^2$
49. $9c^2 - 12cd + 4d^2$
50. $9a^2 - 30ab + 25b^2$
51. In engineering, the equation of transverse shearing stress in a rectangular beam is given by

$$T = \frac{V}{8I}(h^2 - 4v_1^2)$$

Factor the right member.

Review exercises

Factor completely. See sections 4–1, 4–2, 4–3, and 4–4.

1. $x^2 + 8x + 12$
2. $49a^2 - 81$
3. $3ax + bx - 12ay - 4by$
4. $2x^3 + 14x^2 + 24x$
5. $10a^2 + 21a + 9$
6. $4a^2 - 20a + 25$
7. $x^2y^2 + 8xy + 15$
8. $x^2 + 4xy + 4y^2$

4–5 ■ Other types of factoring

The difference of two cubes

In section 4–4, we factored expressions that involved the difference of two squares. To factor these types of expressions, we identified the two terms as perfect squares and applied the procedure. In this section, we will factor the *sum and difference of two cubes* in a similar fashion.

Consider the indicated product of $(a - b)(a^2 + ab + b^2)$. If we carry out the multiplication, we have

$$\begin{aligned}(a - b)(a^2 + ab + b^2) &= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 \\ &= a^3 - b^3\end{aligned}$$

Therefore $(a - b)(a^2 + ab + b^2) = a^3 - b^3$ and $(a - b)(a^2 + ab + b^2)$ is the factored form of $a^3 - b^3$.

The difference of two cubes factors as

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Concept

If we are able to write a two-term polynomial as a first term cubed minus a second term cubed, then it will factor as the difference of two cubes.

$$(1\text{st term})^3 - (2\text{nd term})^3$$
$$= (1\text{st term} - 2\text{nd term})[(1\text{st term})^2 + (1\text{st term})(2\text{nd term}) + (2\text{nd term})^2]$$

To use this factoring technique, we must be able to recognize **perfect cubes.**

Example 4–5 A

Write the following as a quantity cubed, if possible.

1. $27 = 3 \cdot 3 \cdot 3$ — 27 is a perfect cube
   $= (3)^3$ — 27 can be written as 3 cubed
2. $a^3 = a \cdot a \cdot a$ — a is written as a factor three times
   $= (a)^3$ — Writing a^3 as $(a)^3$ shows this is a perfect cube
3. $8a^3 = 2a \cdot 2a \cdot 2a$ — 8 is $2 \cdot 2 \cdot 2$ and a^3 is $a \cdot a \cdot a$
   $= (2a)^3$ — It is now rewritten as a cube
4. $64x^6 = 4x^2 \cdot 4x^2 \cdot 4x^2$ — 64 is $4 \cdot 4 \cdot 4$ and x^6 is $x^2 \cdot x^2 \cdot x^2$
   $= (4x^2)^3$ — It is now rewritten as a cube

▶ ***Quick check*** Write the following as a quantity cubed, if possible. $27y^3$ ■

This is the procedure we use for factoring the difference of two cubes.

Step 1 Identify that we have a perfect cube minus another perfect cube.

Step 2 Rewrite the problem as a first term cubed minus a second term cubed.

$$(1\text{st term})^3 - (2\text{nd term})^3$$

Step 3 Factor the expression into the first term minus the second term, times the first term squared plus the first term times the second term plus the second term squared.

$$(1\text{st term} - 2\text{nd term})[(1\text{st term})^2 + (1\text{st term} \cdot 2\text{nd term}) + (2\text{nd term})^2]$$

Example 4–5 B

Factor completely.

1. $x^3 - 27$ We rewrite x^3 as a cube and 27 as a cube.

   $x^3 - 27 = (x)^3 - (3)^3$

   The first term is x and the second term is 3. Then we write the procedure for factoring the difference of two cubes.

   $$(\underset{\uparrow \atop 1\text{st}}{\quad} - \underset{\uparrow \atop 2\text{nd}}{\quad})[(\underset{\uparrow \atop 1\text{st}}{\quad})^2 + (\underset{\uparrow \atop 1\text{st}}{\quad})(\underset{\uparrow \atop 2\text{nd}}{\quad}) + (\underset{\uparrow \atop 2\text{nd}}{\quad})^2]$$

Now substitute x where the first term is in the procedure and 3 where the second term is.

$$(x - 3)[(x)^2 + (x)(3) + (3)^2]$$
$$\begin{array}{cccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \text{1st} & \text{2nd} & \text{1st} & \text{1st} & \text{2nd} & \text{2nd} \end{array}$$

Finally we simplify.

$$(x - 3)(x^2 + 3x + 9)$$

Therefore $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$.

2. $8x^3 - y^3 = (2x)^3 - (y)^3$ — First term is $2x$ and second term is y.

Then $(\quad - \quad)[(\quad)^2 + (\quad)(\quad) + (\quad)^2]$ — Factoring procedure ready for substitution

$= (2x - y)[(2x)^2 + (2x)(y) + (y)^2]$ — The first term is $2x$, the second term is y

$= (2x - y)(4x^2 + 2xy + y^2)$ — Simplify within the second group

3. $2a^3 - 54b^3 = 2(a^3 - 27b^3)$ — Factor out the common quantity of 2

$2(a^3 - 27b^3) = 2[(a)^3 - (3b)^3]$ — Rewrite as cubes

Then $2(\quad - \quad)[(\quad)^2 + (\quad)(\quad) + (\quad)^2]$ — Factoring procedure ready for substitution

$= 2(a - 3b)[(a)^2 + (a)(3b) + (3b)^2]$ — The first term is a, the second term is $3b$

$= 2(a - 3b)(a^2 + 3ab + 9b^2)$ — Simplify within the second group

4. $a^{15} - 64b^3 = (a^5)^3 - (4b)^3$ — Rewrite as cubes

Then $(\quad - \quad)[(\quad)^2 + (\quad)(\quad) + (\quad)^2]$ — Factoring procedure ready for substitution

$= (a^5 - 4b)[(a^5)^2 + (a^5)(4b) + (4b)^2]$ — The first term is a^5, the second term is $4b$

$= (a^5 - 4b)(a^{10} + 4a^5b + 16b^2)$ — Simplify within the second group

Note In example 4, we observe that a number raised to a power that is a multiple of 3 can be written as a cube by dividing the exponent by 3. The quotient is the exponent of the number inside the parentheses and the 3 is the exponent outside the parentheses. For example, $y^{12} = (y^4)^3$ or $z^{24} = (z^8)^3$.

▶ ***Quick check*** Factor $16R^3 - 54$

The sum of two cubes

If we carry out the indicated multiplication in $(a + b)(a^2 - ab + b^2)$, we have

$$\begin{aligned}(a + b)(a^2 - ab + b^2) &= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 \\ &= a^3 + b^3\end{aligned}$$

Therefore $(a + b)(a^2 - ab + b^2) = a^3 + b^3$ and $(a + b)(a^2 - ab + b^2)$ is the factored form of $a^3 + b^3$.

The sum of two cubes factors as

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Concept

If we are able to write a two-term polynomial as a first term cubed plus a second term cubed, then it will factor as the sum of two cubes.

$$(\text{1st term})^3 + (\text{2nd term})^3$$
$$= (\text{1st term} + \text{2nd term})[(\text{1st term})^2 - (\text{1st term})(\text{2nd term}) + (\text{2nd term})^2]$$

This is the procedure we use for factoring the sum of two cubes.

Step 1 Identify that we have a perfect cube plus another perfect cube.

Step 2 Rewrite the problem as a first term cubed plus a second term cubed.

$$(\text{1st term})^3 + (\text{2nd term})^3$$

Step 3 Factor the expression into the first term plus the second term, times the first term squared minus the first term times the second term plus the second term squared.

$$(\text{1st term} + \text{2nd term})[(\text{1st term})^2 - (\text{1st term})(\text{2nd term}) + (\text{2nd term})^2]$$

Example 4–5 C

Factor completely.

1. $a^3 + 8 = (a)^3 + (2)^3$

We now write the procedure for the sum of two cubes.

$$(\underset{\substack{\uparrow\\ \text{1st}}}{\quad} + \underset{\substack{\uparrow\\ \text{2nd}}}{\quad})[(\underset{\substack{\uparrow\\ \text{1st}}}{\quad})^2 - (\underset{\substack{\uparrow\\ \text{1st}}}{\quad})(\underset{\substack{\uparrow\\ \text{2nd}}}{\quad}) + (\underset{\substack{\uparrow\\ \text{2nd}}}{\quad})^2]$$

Substituting,

$$(\underset{\substack{\uparrow\\ \text{1st}}}{a} + \underset{\substack{\uparrow\\ \text{2nd}}}{2})[(\underset{\substack{\uparrow\\ \text{1st}}}{a})^2 - (\underset{\substack{\uparrow\\ \text{1st}}}{a})\,(\underset{\substack{\uparrow\\ \text{2nd}}}{2}) + (\underset{\substack{\uparrow\\ \text{2nd}}}{2})^2]$$

simplifying, $(a + 2)(a^2 - 2a + 4)$

therefore $a^3 + 8 = (a + 2)(a^2 - 2a + 4)$

2. $x^3 + 125 = (x)^3 + (5)^3$ — Rewrite as cubes

Then $(\quad + \quad)[(\quad)^2 - (\quad)(\quad) + (\quad)^2]$ — Factoring procedure ready for substitution

$= (x + 5)[(x)^2 - (x)(5) + (5)^2]$ — The first term is x, the second term is 5

$= (x + 5)(x^2 - 5x + 25)$ — Simplify within the second group

3. $8a^3 + b^{21} = (2a)^3 + (b^7)^3$ — Rewrite as cubes

Then $(\quad + \quad)[(\quad)^2 - (\quad)(\quad) + (\quad)^2]$ — Factoring procedure ready for substitution

$= (2a + b^7)[(2a)^2 - (2a)(b^7) + (b^7)^2]$ — The first term is $2a$, the second term is b^7

$= (2a + b^7)(4a^2 - 2ab^7 + b^{14})$ — Simplify within the second group

4. $x^3y^3 + z^3 = (xy)^3 + (z)^3$ — Rewrite as cubes

Then $(\quad + \quad)[(\quad)^2 - (\quad)(\quad) + (\quad)^2]$ — Factoring procedure ready for substitution

$= (xy + z)[(xy)^2 - (xy)(z) + (z)^2]$ — The first term is xy, the second term is z

$= (xy + z)(x^2y^2 - xyz + z^2)$ — Simplify within the second group

▶ ***Quick check*** Factor $27x^3 + y^3$

Mastery points

Can you

- Identify and rewrite a perfect cube?
- Factor the sum or difference of two cubes?

Exercise 4–5

Write the following as a quantity cubed, if possible. See example 4–5 A.

Example $27y^3$

Solution $= 3y \cdot 3y \cdot 3y$ — 27 is $3 \cdot 3 \cdot 3$ and y^3 is $y \cdot y \cdot y$
$= (3y)^3$ — It is now rewritten as a cube

1. 64 **2.** 8 **3.** 125 **4.** 1 **5.** $27x^3$
6. $64a^3$ **7.** a^6 **8.** x^9 **9.** $8b^{15}$ **10.** $64c^{21}$

Factor completely. If an expression will not factor, so state. See examples 4–5 B and C.

Example $16R^3 - 54$

Solution
$= 2[8R^3 - 27]$ — Common factor of 2
$= 2[(2R)^3 - (3)^3]$ — Rewrite as cubes
$2(\quad - \quad)[(\quad)^2 + (\quad)(\quad) + (\quad)^2]$ — Factoring procedure ready for substitution
$= 2(2R - 3)[(2R)^2 + (2R)(3) + (3)^2]$ — The first term is $2R$, the second term is 3
$= 2(2R - 3)(4R^2 + 6R + 9)$ — Simplify within the second group

Example $27x^3 + y^3$

Solution
$= (3x)^3 + (y)^3$ — Rewrite as cubes
$(\quad + \quad)[(\quad)^2 - (\quad)(\quad) + (\quad)^2]$ — Factoring procedure ready for substitution
$= (3x + y)[(3x)^2 - (3x)(y) + (y)^2]$ — The first term is $3x$, the second term is y
$= (3x + y)(9x^2 - 3xy + y^2)$ — Simplify within the second group

11. $r^3 + s^3$ **12.** $L^3 + 8$ **13.** $8x^3 + y^3$ **14.** $27r^3 + 8$
15. $h^3 - k^3$ **16.** $p^3 - q^3$ **17.** $a^3 - 8$ **18.** $b^3 + 64$
19. $x^3 - 8y^3$ **20.** $27a^3 - b^3$ **21.** $64x^3 - y^3$ **22.** $r^3 - 27$
23. $27x^3 - 8y^3$ **24.** $64a^3 - 8$ **25.** $8a^3 + 27b^3$ **26.** $64s^3 + 1$
27. $2a^3 + 16$ **28.** $3x^3 + 81$ **29.** $2x^3 - 16$ **30.** $81a^3 - 3b^3$
31. $x^5 + 27x^2y^3$ **32.** $16a^3 + 2b^3$ **33.** $x^6 + y^3$ **34.** $x^3 + y^9$
35. $a^9 - b^3$ **36.** $a^6 - 8$ **37.** $x^{12} - 27$ **38.** $x^{15} + 64$
39. $8a^2b^3 - a^5$ **40.** $2x^3 - 54y^3$ **41.** $54r^3 + 2s^3$ **42.** $b^5 + 64b^2c^3$
43. $x^3y^3 - z^3$ **44.** $x^3y^9 - 1$ **45.** $a^{15}b^6 - 8c^9$ **46.** $x^{18}y^9 - 27z^3$
47. $a^3b^3 + 8$ **48.** $x^3y^6 + z^3$ **49.** $x^9y^{12} + z^{15}$ **50.** $a^{12}b^{15} + c^{24}$

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Review exercises

Write in completely factored form. See sections 4–1, 4–2, 4–3, and 4–4.

1. $a^2 - 7a + 10$ **2.** $6ax + 2bx - 3ay - by$ **3.** $x^2 + 4xy + 4y^2$

4. $6a^2 - ab - b^2$ **5.** $5a^3 - 40a^2 + 75a$ **6.** $6x^2 + x - 12$

4–6 ■ Factoring: A general strategy

In this section, we will review the different methods of factoring that we have studied in the previous sections. The following outline gives a general strategy for factoring polynomials.

I. Factor out any common factors.
Examples
1. $5a^3 - 25a^2 = 5a^2(a - 5)$
2. $c(a - 2b) + 2d(a - 2b) = (a - 2b)(c + 2d)$

II. Count the number of terms.

A. Two terms: Check to see if the polynomial is the difference of two squares, the difference of two cubes, or the sum of two cubes.
Examples
1. $a^2 - 16b^2 = (a - 4b)(a + 4b)$ Difference of two squares
2. $8a^3 - b^3 = (2a - b)(4a^2 + 2ab + b^2)$ Difference of two cubes
3. $m^3 + 64n^3 = (m + 4n)(m^2 - 4mn + 16n^2)$ Sum of two cubes

B. Three terms: Check to see if the polynomial is a perfect square trinomial. If it is not, use one of the general methods for factoring a trinomial.
Examples
1. $a^2 + 6a + 9 = (a + 3)^2$ Perfect square trinomial
2. $a^2 + 5a - 14 = (a + 7)(a - 2)$ General trinomial, leading coefficient of 1
3. $6a^2 + 7a - 20 = (2a + 5)(3a - 4)$ General trinomial, leading coefficient other than 1

C. Four terms: Check to see if we can factor by grouping.
Examples
1. $ac + 3a - 2bc - 6b = (a - 2b)(c + 3)$
2. $a^3 + 2a^2 - 3a - 6 = (a^2 - 3)(a + 2)$

III. Check to see if any of the factors we have written can be factored further. Any common factors that were missed in part I can still be factored out here.
Examples
1. $c^4 - 11c^2 + 28 = (c^2 - 4)(c^2 - 7)$
$= (c - 2)(c + 2)(c^2 - 7)$ Difference of two squares
2. $4a^2 - 36b^2 = (2a - 6b)(2a + 6b)$
$= 2(a - 3b)2(a + 3b)$
$= 4(a - 3b)(a + 3b)$ Overlooked common factor

The following examples illustrate our strategy for factoring polynomials.

■ Example 4–6 A

Completely factor the following polynomials.

1. $3x^3 - 3xy^2$

I. First we look for any common factors.

$$3x^3 - 3xy^2 = 3x(x^2 - y^2) \quad \text{Common factor of } 3x$$

II. The factor $x^2 - y^2$ has two terms and is the difference of two squares.

$$x^2 - y^2 = (x - y)(x + y) \quad \text{Factoring the binomial}$$

III. After checking to see if any of the factors will factor further, we conclude that $3x(x - y)(x + y)$ is the completely factored form. Therefore

$$3x^3 - 3xy^2 = 3x(x - y)(x + y)$$

2. $3ax + bx + 6ay + 2by$

I. There is no common factor (other than 1 or -1).

II. The polynomial has four terms and we factor it by grouping.

$$\begin{aligned} &(3ax + bx) + (6ay + 2by) && \text{Group in pairs} \\ &\quad = x(3a + b) + 2y(3a + b) && \text{Factor out what is common to each pair} \\ &\quad = (3a + b)(x + 2y) && \text{Factor out the common quantity} \end{aligned}$$

III. None of the factors will factor further.

$$3ax + bx + 6ay + 2by = (3a + b)(x + 2y)$$

3. $3a^2 - 2a - 8$

I. There is no common factor (other than 1 or -1).

II. The polynomial has three terms and the coefficient of a^2 is not 1. Therefore we must find m and n and factor the trinomial. $m + n = -2$ and $m \cdot n = -24$, the values for m and n are -6 and 4.

$$\begin{aligned} &= 3a^2 - 6a + 4a - 8 && \text{Replace } -2a \text{ with } -6a + 4a \\ &= (3a^2 - 6a) + (4a - 8) && \text{Group the first two terms and the last two terms} \\ &= 3a(a - 2) + 4(a - 2) && \text{Factor out what is common to each pair} \\ &= (a - 2)(3a + 4) && \text{Factor out the common quantity} \end{aligned}$$

III. None of the factors will factor further.

$$3a^2 - 2a - 8 = (a - 2)(3a + 4)$$

▶ ***Quick check*** Factor $4x^2 - 36y^2$ ■

Mastery points

Can you

- Factor out the greatest common factor?
- Factor the difference of two squares?
- Factor the difference of two cubes?
- Factor the sum of two cubes?
- Factor trinomials?
- Factor a four-term polynomial?
- Use the general strategy for factoring polynomials?

Exercise 4–6

Completely factor the following polynomials. If a polynomial will not factor, so state. See the outline of the general strategy for factoring polynomials and example 4–6 A.

Example $4x^2 - 36y^2$

Solution

I.	$4x^2 - 36y^2 = 4(x^2 - 9y^2)$	Common factor of 4
II.	$= 4(x - 3y)(x + 3y)$	Factoring the difference of two squares
III.	$4x^2 - 36y^2 = 4(x - 3y)(x + 3y)$	Completely factored form

1. $n^2 - 49$
2. $a^2 + 6a + 5$
3. $7b^2 + 36b + 5$
4. $2x^2 + 15x + 18$
5. $x^2y^2 + 2xy - 8$
6. $y^2 + 11y + 10$
7. $36 - y^2$
8. $25a^2(3b + c) + 5a(3b + c)$
9. $10a^2 - 20ab + 10b^2$
10. $a^2b^2 - 5ab - 14$
11. $4a^2 - 16b^2$
12. $12x^3y^2 - 18x^2y^2 + 16xy^4$
13. $3ax + 6ay - bx - 2by$
14. $5x^2 + 18x - 60$
15. $6x^2 + 7x - 5$
16. $9x^5y - 6x^3y^3 + 3x^2y^2$
17. $6am + 4bm - 3an - 2bn$
18. $5x^2 - 32x - 21$
19. $7b^2 + 16b - 15$
20. $3a^2 + 13a + 4$
21. $4x^2 + 17x - 15$
22. $5y^2 + 16y + 12$
23. $6x^2 - 24xy - 48y^2$
24. $4ab(x + 3y) - 8a^2b^2(x + 3y)$
25. $3x^2y(m - 4n) + 15xy^2(m - 4n)$
26. $4x^2 - 20xy + 25y^2$
27. $9a^2 - 30ab + 25b^2$
28. $80y^4 - 5y$
29. $3a^5 - 48a$
30. $3a^5b - 18a^3b^3 + 27ab^5$
31. $3a^3b^3 + 6a^2b^4 + 3ab^5$
32. $3b^2 + 8b - 91$
33. $3b^2 - 32b - 91$
34. $b^4 - 81$
35. $3ax + 6bx + 2ay + 4by$
36. $12ax + 4bx - 3ay - by$
37. $6x^2 + 11x - 2$
38. $6x^2 - 17x - 3$
39. $3x^4 - 48x^2$
40. $3x^3 + 3x^2 - 18x$
41. $y^3 + 27z^3$
42. $8b^3 - c^3$
43. $x^3 - y^9$
44. $a^3b^3 + 64$

Review exercises

Find the solution set of the following equations. See section 2–6.

1. $2x + 6 = 0$
2. $4x - 12 = 0$
3. $3x - 18 = 0$
4. $5x + 3 = 0$
5. $6x + 4 = 0$
6. $3x + 1 = 0$
7. $4x - 1 = 0$
8. $3x = 0$

4–7 ■ Solving quadratic equations by factoring

The standard form of a quadratic equation

In chapter 2, we studied linear equations, also called first-degree equations. Recall that the **degree** of an equation in one variable is the greatest exponent of that variable in any one term. In this section, we will find the solutions to an equation that contains the second, but no higher, power of that variable. Such an equation is a **second-degree equation,** also called a **quadratic equation.**

Quadratic equations can be written in the form

$$ax^2 + bx + c = 0$$

where a, b, and c are constants, $a \neq 0$.

Note It is necessary that $a \neq 0$. If $a = 0$ and $b \neq 0$, then $0 \cdot x^2 + bx + c$ becomes $bx + c = 0$, which is a linear equation.

We call $ax^2 + bx + c = 0$ where $a > 0$ the **standard quadratic form** of a quadratic equation. Notice that in standard form the terms of the left member of the equation are written in descending powers of the variable. The right member contains *only zero*. In most of our work with quadratic equations, it will be necessary to write the equation in standard quadratic form.

Solution of quadratic equations in factored form

Now suppose we have the equation $x^2 - x - 6 = 0$ stated in the factored form, $(x - 3)(x + 2) = 0$. This equation states that the product of two factors, $x - 3$ and $x + 2$, is 0. To find the necessary numbers, we use the algebraic property called the **zero product property.**

Zero product property

Given real numbers p and q, if $pq = 0$, then $p = 0$ or $q = 0$.

Concept

If the product of two factors is zero, then at least one of the factors is zero.

Extending this property, if $(x + p)(x + q) = 0$, then $x + p = 0$ or $x + q = 0$. Therefore by this property, $(x - 3)(x + 2) = 0$ only if $x - 3 = 0$ or $x + 2 = 0$. Since each of these equations is linear, we use the methods for solving linear equations. We find $x = 3$ when $x - 3 = 0$ and $x = -2$ when $x + 2 = 0$. Then 3 and -2 are solutions of the equation $(x - 3)(x + 2) = 0$.

From this discussion, we can see that to solve any equation in *factored form* whose product is 0, we solve an equation of the form $p \cdot q = 0$.

Solving an equation of the form $p \cdot q = 0$

1. Set *each* factor equal to 0.
2. Solve the resulting equations for the variable.

Solution set

In the equation $x^2 - x - 6 = 0$, we saw that the solutions of the equation, $(x - 3)(x + 2) = 0$, were 3 and -2. To express this as a solution set, we would write the solutions in any order, separated by a comma and enclosed within a pair of braces. The solution set for the equation is $\{-2,3\}$.

Example 4–7 A

Find the solution set of the following equations.

1. $(x + 5)(x - 4) = 0$ — The equation is in factored form

$x + 5 = 0$ or $x - 4 = 0$ — Set each factor equal to 0 and solve

$x = -5$ $\quad$ $x = 4$ — The solutions

Check for $x = -5$ $\qquad$ Check for $x = 4$

$(-5 + 5)(-5 - 4) = 0$ $\quad$ $(4 + 5)(4 - 4) = 0$ — Substitute the solution for x

$0 \cdot (-9) = 0$ $\quad$ $9 \cdot 0 = 0$ — Order of operations

$0 = 0$ $\quad$ $0 = 0$ — True, both solutions check

The solution set is $\{-5,4\}$.

Note In future examples, we will not always show a check of the solutions, but checking your solutions is always an important part of the problem.

2. $(x - 3)(3x + 1) = 0$ — The equation is in factored form

$x - 3 = 0$ or $3x + 1 = 0$ — Set each factor equal to 0 and solve

$x = 3$ $\quad$ $3x = -1$ — Add 3, subtract 1

$x = -\frac{1}{3}$ — Divide by 3

$x = 3$ or $x = -\frac{1}{3}$ — The solutions

The solution set is $\left\{-\frac{1}{3},3\right\}$.

3. $3x(x - 7) = 0$ — The equation is in factored form

$3x = 0$ or $x - 7 = 0$ — Set each factor equal to 0 and solve

$x = 0$ $\quad$ $x = 7$ — Dividing by 3, adding 7 gives the solutions

The solution set is $\{0,7\}$.

▶ ***Quick check*** Find the solution set of $(2x + 3)(x + 1) = 0$

Solving quadratic equations by factoring

In general, to find the solution set of a quadratic equation by factoring, we use the following procedure:

Solving a quadratic equation by factoring

Step 1 Write the equation in standard quadratic form.
Step 2 Completely factor the left member.
Step 3 Set each of the factors containing the variable equal to 0 and solve the resulting equations.
Step 4 Write the solutions in a solution set.
Step 5 Check your solutions by substituting into the original equation.

Example 4–7 B

Find the solution set of the following equations. Check your solutions.

1.

$$x^2 + 5x = -6$$

$$x^2 + 5x + 6 = 0 \qquad \text{Write the equation in standard form}$$

$$(x + 2)(x + 3) = 0 \qquad \text{Factor } x^2 + 5x + 6 = (x + 2)(x + 3)$$

$$x + 2 = 0 \quad \text{or} \quad x + 3 = 0 \qquad \text{Set each factor equal to 0}$$

$$x = -2 \qquad\qquad x = -3 \qquad \text{Solve each equation, giving the solutions}$$

Check:

(1) Let $x = -2$

$$x^2 + 5x = -6$$

$$(-2)^2 + 5(-2) = -6$$

$$4 - 10 = -6$$

$$-6 = -6 \quad \text{True}$$

(2) Let $x = -3$

$$x^2 + 5x = -6$$

$$(-3)^2 + 5(-3) = -6$$

$$9 - 15 = -6$$

$$-6 = -6 \quad \text{True}$$

The solution set is $\{-3,-2\}$.

2.

$$x^2 = 2x$$

$$x^2 - 2x = 0 \qquad \text{Write the equation in standard form}$$

$$x(x - 2) = 0 \qquad \text{Factor completely}$$

$$x = 0 \quad \text{or} \quad x - 2 = 0 \qquad \text{Set each factor equal to 0}$$

$$x = 0 \qquad\qquad x = 2 \qquad \text{Solve each equation, giving the solutions}$$

Check your solutions by substituting 0 and 2 for x in the original equation.

The solution set is $\{0,2\}$.

3.

$$x^2 = 16$$

$$x^2 - 16 = 0 \qquad \text{Write the equation in standard form.}$$

$$(x - 4)(x + 4) = 0 \qquad \text{Factor } x^2 - 16 = (x - 4)(x + 4)$$

$$x - 4 = 0 \quad \text{or} \quad x + 4 = 0 \qquad \text{Set each factor equal to 0}$$

$$x = 4 \qquad\qquad x = -4 \qquad \text{Solve each equation, giving the solutions}$$

Check your solutions by substituting 4 and -4 for x in the original equation.

The solution set is $\{-4,4\}$.

4.

$$4x^2 = 20x - 25$$

$$4x^2 - 20x + 25 = 0 \qquad \text{Write the equation in standard form}$$

$$(2x - 5)^2 = 0 \qquad 4x^2 - 20x + 25 = (2x - 5)^2$$

$$2x - 5 = 0 \qquad \text{Set the repeated factor equal to 0}$$

$$x = \frac{5}{2} \qquad \text{Solve the equation for } x\text{, giving the solution}$$

Check your solution by substituting $\frac{5}{2}$ for x in the original equation.

The solution set is $\left\{\frac{5}{2}\right\}$.

Note In example 4, we have *two* factors, $(2x - 5)$ and $(2x - 5)$, but since they are the same, the equation has only *one* distinct solution.

5.
$$
\begin{aligned}
3x^2 + 3 &= -6x \\
3x^2 + 6x + 3 &= 0 && \text{Write the equation in standard form} \\
3(x^2 + 2x + 1) &= 0 && \text{Factor } 3x^2 + 6x + 3 = 3(x+1)^2 \\
3(x+1)^2 &= 0 && \text{Set the only distinct factor with a variable equal to 0} \\
x + 1 &= 0 \\
x &= -1 && \text{Solve the equation, giving the solution}
\end{aligned}
$$

Check your solution by substituting -1 for x in $3x^2 + 3 = -6x$.

The solution set is $\{-1\}$.

▶ ***Quick check*** Find the solution set of each of the following: $2a^2 = 5a$, $4y^2 = 9$, $x^2 - 7x + 12 = 0$, and $12(x^2 - 2x) = -9$ ■

In conclusion, let us compare the quadratic equation with the linear equation.

1. A *linear equation* is an equation of the form $ax + b = 0$, where $a \neq 0$. A *quadratic equation* is an equation of the form $ax^2 + bx + c = 0$, where a, b, and c are constants, $a > 0$.
2. A linear equation is solved by isolating the variable. Some quadratic equations are solved by factoring and setting the linear factors equal to zero.
3. A conditional linear equation has at most *one solution.* A quadratic equation has at most *two real solutions,* which may or may not be distinct.

Mastery points

Can you

- Find the solution set of an equation in factored form whose product is equal to zero?
- Find the solution set of a quadratic equation by factoring?

Exercise 4–7

Find the solution set of the following equations. See example 4–7 A.

Example $(2x + 3)(x + 1) = 0$ — The equation is in factored form

Solution Set each factor equal to 0 and solve the equations.

$$
\begin{aligned}
2x + 3 &= 0 \quad \text{or} \quad x + 1 = 0 && \text{The factors are set equal to 0} \\
2x &= -3 && \text{Solve each equation} \\
x &= -\frac{3}{2} \qquad\qquad x = -1 && \text{The solutions}
\end{aligned}
$$

The solution set is $\left\{-\frac{3}{2}, -1\right\}$.

$\boxed{1.}$ $(x + 5)(x - 5) = 0$ **2.** $(x - 1)(x + 1) = 0$ **3.** $x(x + 6) = 0$

4. $x(x - 8) = 0$ **5.** $3a(a - 7) = 0$ **6.** $5p(p + 9) = 0$

$\boxed{7.}$ $(3x - 9)(2x + 3) = 0$
8. $(8x - 4)(5x + 10) = 0$
9. $(2x + 1)(3x - 2) = 0$
10. $(4y - 3)(5y + 2) = 0$
11. $(5x - 1)(5x + 1) = 0$
12. $(4a - 1)(4a + 1) = 0$
13. $(8 - y)(7 - y) = 0$
14. $(1 - x)(3 - x) = 0$
15. $(4 - 3u)(8 - 5u) = 0$
16. $(7 + 3y)(2 - 3y) = 0$
17. $(5 - x)(5 + x) = 0$
18. $(8 - x)(8 + x) = 0$
19. $x(2x - 3)(x + 1) = 0$
20. $3x(x - 12)(3x + 1) = 0$
21. $(5x + 3)(x - 10)(4x - 1) = 0$
22. $(8x - 1)(2x + 7)(x - 3) = 0$

Find the solution set of the following quadratic equations by factoring. Check the solutions. See example 4–7 B.

Example $2a^2 = 5a$

Solution

$2a^2 - 5a = 0$	Write the equation in standard form
$a(2a - 5) = 0$	Factor $2a^2 - 5a = a(2a - 5)$
$a = 0$ or $2a - 5 = 0$	Set each factor equal to 0
$a = 0 \qquad 2a = 5$	Solve each equation
$a = 0 \qquad a = \dfrac{5}{2}$	The solutions

Check:

(1) Let $a = 0$

$2a^2 = 5a$

$2(0)^2 = 5(0)$

$0 = 0$ (True) Checks

(2) Let $a = \dfrac{5}{2}$

$2\left(\dfrac{5}{2}\right)^2 = 5\left(\dfrac{5}{2}\right)$ Substitute

$2\left(\dfrac{25}{4}\right) = \dfrac{25}{2}$ Order of operations

$\dfrac{25}{2} = \dfrac{25}{2}$ (True) Checks

The solution set is $\left\{0, \dfrac{5}{2}\right\}$.

Example $4y^2 = 9$

Solution

$4y^2 - 9 = 0$	Write the equation in standard form
$(2y + 3)(2y - 3) = 0$	Factor $4y^2 - 9 = (2y + 3)(2y - 3)$
$2y + 3 = 0$ or $2y - 3 = 0$	Set each factor equal to 0
$2y = -3 \qquad 2y = 3$	Solve each equation
$y = -\dfrac{3}{2} \qquad y = \dfrac{3}{2}$	The solutions

Check by replacing y with $-\dfrac{3}{2}$ and $\dfrac{3}{2}$ in the original equation.

The solution set is $\left\{-\dfrac{3}{2}, \dfrac{3}{2}\right\}$.

23. $x^2 + 4x = 0$
24. $y^2 - 4y = 0$
25. $3a^2 - 5a = 0$
26. $4x^2 + 7x = 0$
27. $2x^2 + 6x = 0$
28. $3b^2 = 9b$
29. $2y^2 - 18 = 0$
30. $6x^2 = 24x$

$\boxed{31.}$ $10a^2 = -15a$ **32.** $4y^2 = -6y$ **33.** $a^2 = 25$ **34.** $y^2 = 49$
35. $x^2 - 9 = 0$ **36.** $y^2 - 100 = 0$ **37.** $2x^2 = 32$ **38.** $3a^2 = 27$
39. $7x^2 - 28 = 0$ $\boxed{40.}$ $5y^2 - 45 = 0$ **41.** $8x^2 - 18 = 0$ **42.** $7x^2 - 7 = 0$

Find the solution set of the following quadratic equations by factoring. Check the solutions. See example 4–7 B.

Example $x^2 - 7x + 12 = 0$

Solution

$x^2 - 7x + 12 = 0$ The equation is in standard form
$(x - 3)(x - 4) = 0$ Factor $x^2 - 7x + 12 = (x - 3)(x - 4)$
$x - 3 = 0$ or $x - 4 = 0$ Set each factor equal to 0
$x = 3$ $x = 4$ Solve each equation, giving the solutions

Check:

(1) Let $x = 3$
$$x^2 - 7x + 12 = 0$$
$$(3)^2 - 7(3) + 12 = 0$$
$$9 - 21 + 12 = 0$$
$$-12 + 12 = 0$$
$$0 = 0 \quad \text{(True) Checks}$$

(2) Let $x = 4$
$$x^2 - 7x + 12 = 0$$
$$(4)^2 - 7(4) + 12 = 0 \quad \text{Substitute}$$
$$16 - 28 + 12 = 0 \quad \text{Order of operations}$$
$$-12 + 12 = 0$$
$$0 = 0 \quad \text{(True) Checks}$$

The solution set is $\{3,4\}$.

43. $y^2 + 6y - 16 = 0$ **44.** $x^2 - 3x - 4 = 0$ **45.** $a^2 + 14a + 49 = 0$
46. $x^2 - 16x + 64 = 0$ $\boxed{47.}$ $b^2 + 5b - 14 = 0$ **48.** $x^2 + x - 42 = 0$
49. $x^2 + 3x + 2 = 0$ **50.** $y^2 = -11y - 10$ **51.** $a^2 - 11a = 12$
$\boxed{52.}$ $x^2 - 14x = 15$ **53.** $y^2 - 32 = 4y$ **54.** $x^2 = 27 - 6x$

Example $3x^2 = 7x + 6$

Solution

$3x^2 - 7x - 6 = 0$ Write in standard form
$(3x + 2)(x - 3) = 0$ Factor $3x^2 - 7x - 6 = (3x + 2)(x - 3)$
$3x + 2 = 0$ or $x - 3 = 0$ Set each factor equal to 0
$3x = -2$ $x = 3$ Solve the equations
$x = -\dfrac{2}{3}$ $x = 3$ The solutions

Check by replacing x with $-\dfrac{2}{3}$ and 3 in the original equation.

The solution set is $\left\{-\dfrac{2}{3},3\right\}$.

55. $2x^2 - 7x - 9 = 0$ **56.** $2y^2 - y - 3 = 0$ **57.** $6a^2 - 5a + 1 = 0$
58. $3p^2 + 10p - 8 = 0$ $\boxed{59.}$ $6x^2 + x - 12 = 0$ **60.** $12x^2 - 25x + 12 = 0$
61. $4y^2 = 4y + 3$ **62.** $6a^2 + 3 = 11a$ **63.** $9x^2 = 8x + 1$
$\boxed{64.}$ $-6x = -3x^2 - 3$ **65.** $6p^2 - 7p = 20$ **66.** $9y^2 + 20 = -27y$
67. $3x^2 + 6x = -3$ **68.** $15 - 2x^2 = -x$ **69.** $9x^2 + 30x = -25$
70. $12y^2 - 15 = 8y$ $\boxed{71.}$ $3x^2 - 4x - 28 = x$ **72.** $6z^2 + z = -10z - 3$
73. $2b^2 + 2b = 9 - b$ **74.** $3a - 3 = 15a^2 + 2a - 5$

Example $12(x^2 - 2x) = -9$

Solution

$$12x^2 - 24x = -9 \quad \text{Multiply in the left member}$$
$$12x^2 - 24x + 9 = 0 \quad \text{Write the equation in standard form}$$
$$3(4x^2 - 8x + 3) = 0 \quad \text{Factor common factor 3}$$
$$3(2x - 1)(2x - 3) = 0 \quad \text{Factor } 4x^2 - 8x + 3 = (2x - 1)(2x - 3)$$
$$2x - 1 = 0 \quad \text{or} \quad 2x - 3 = 0 \quad \text{Set each factor containing the variable equal to 0}$$

Note The constant factor 3 cannot be zero, so we disregard it when finding the solutions.

$$2x = 1 \quad \text{or} \quad 2x = 3 \quad \text{Solve each equation}$$
$$x = \frac{1}{2} \qquad x = \frac{3}{2} \quad \text{The solutions}$$

Check by replacing x with $\frac{1}{2}$ and $\frac{3}{2}$ in the original equation.

The solution set is $\left\{\frac{1}{2}, \frac{3}{2}\right\}$.

75. $x(x + 3) = -2$
$\boxed{76.}$ $3x(3x + 2) = 24$
77. $2x(2x + 6) = -8$
78. $2(x^2 - 6) = -5x$
79. $x(x - 6) = 18 + x$
80. $x(x + 1) - 6 = 0$
81. $4 = 3x(4 - 3x)$
82. $2x(3 - x) = 4$
83. $5(x^2 - 5) = 20x$
$\boxed{84.}$ $x(x + 7) = 36 - 2x$

Review exercises

Write an algebraic expression for each of the following. See section 2–1.

1. 7 more than a number
2. A number decreased by 11
3. 6 times the sum of x^2 and x
4. The sum of x^2 and $2x$, divided by 8

Solve the following word problems. See section 2–8.

5. Three times a number is increased by 12 and the result is 51. Find the number.
6. One number is two more than five times another number. If their sum is 38, find the numbers.
7. The sum of three consecutive even integers is 72. Find the integers.

4–8 ■ Applications of the quadratic equation

Many formulas used in the physical world are quadratic in nature since they become second-degree equations when solving for one of the variables. Likewise, many word problems require the use of quadratic equations for their solutions. We now consider some of these common uses of the quadratic equation.

Be aware that a check of the solutions by merely substituting them into the equation set up to solve the problem *will not* guarantee the correct answer. We should check the results to ensure that the physical conditions of the word problem are satisfied.

Example 4–8 A

1. The product of two consecutive even integers is 168. Find the integers.

   Let x = the lesser even integer. Then $x + 2$ = the next consecutive even integer.

Note Consecutive even or odd integers are given by $x, x + 2, x + 4, \cdots$.

two consecutive even integers	product of		is	168
x	$\cdot$	$(x + 2)$	$=$	168

$x^2 + 2x = 168$	Original equation
$x^2 + 2x - 168 = 0$	Write in standard form
$(x + 14)(x - 12) = 0$	Factor the left member
$x + 14 = 0$ or $x - 12 = 0$	Set each factor equal to zero
$x = -14$ $\qquad$ $x = 12$	Solve each equation

When $x = -14$, then $x + 2 = -14 + 2 = -12$.
When $x = 12$, then $x + 2 = 12 + 2 = 14$.
Check: Since $(-14)(-12) = 168$ and $(12)(14) = 168$, and both solutions are consecutive even integers, the conditions of the problem are met.
Therefore, the two integers are -14 and -12 or 12 and 14.

2. The area of a rectangle is $A = \ell w$, where ℓ is the length and w is the width of the rectangle. The length of a rectangle is 2 inches more than three times the width. If $A = 33$ square inches, find the length and width of the rectangle.

   Let w = the width of the rectangle. Length $\ell = 3w + 2$

area of a rectangle	is	length	times	width
A	$=$	ℓ	$\cdot$	w
33	$=$	$(3w + 2)$	$\cdot$	w

$w(3w + 2) = 33$	Original equation
$3w^2 + 2w = 33$	Distribute multiplication
$3w^2 + 2w - 33 = 0$	Write in standard form
$(3w + 11)(w - 3) = 0$	Factor the left member
$3w + 11 = 0$ or $w - 3 = 0$	Set each factor equal to zero
$3w = -11$ $\qquad$ $w = 3$	Solve each equation
$w = -\frac{11}{3}$ $\qquad$ $w = 3$	Solutions of the equation

The solution set of the equation we wrote is $\left\{-\frac{11}{3}, 3\right\}$. Since the width cannot be negative, $w = -\frac{11}{3}$ is not a solution of the problem, even though it *is* a solution of the equation. So $w = 3$ is the only physical solution and

$\ell = 3w + 2 = 3(3) + 2 = 9 + 2 = 11$

Check: Since 11 is two more than 3 times 3 and $(3)(11) = 33$, the conditions of the problem are met.
The rectangle is 3 inches wide and 11 inches long.

3. Current in a circuit flows according to the equation $i = 16 - 16t^2$, where i is the current in amperes and t is the time in seconds. Find the time t when $i = 0$ amperes (no current).

Replacing i by 0, we have the equation

$0 = 16 - 16t^2$	Substitute 0 for i
$0 = 16(1 - t^2)$	Factor the common factor 16
$0 = 16(1 - t)(1 + t)$	Factor $1 - t^2 = (1 - t)(1 + t)$
$1 - t = 0$ or $1 + t = 0$	Set each factor containing the variable equal to 0
$t = 1$ $t = -1$	Solutions of the equation

Since time cannot be negative, $t = 1$ second.

Mastery points

Can you

- Set up a quadratic equation for a word problem and solve the equation?
- Substitute values into a formula and solve the resulting quadratic equation?

Exercise 4–8

Solve the following word problems by setting up a quadratic equation. See example 4–8 A–1.

1. The product of two consecutive odd integers is 143. Find the integers.
2. The product of two consecutive integers is 132. Find the integers.
3. The product of two consecutive even integers is 288. Find the integers.
4. The product of two consecutive integers is 306. Find the integers.
5. One integer is six more than a second integer. The product of the two integers is 91. Find the integers.
6. One integer is eight less than a second integer. The product of the two integers is 153. Find the integers.
7. The product of two consecutive even integers is four more than two times their sum. Find the integers.
8. The product of two consecutive odd integers is five more than six times the lesser integer. Find the integers.
9. The sum of two integers is -13 and their product is 36. Find the integers.
10. The sum of two integers is -3 and their product is -70. Find the integers.
11. One number is one more than three times the other. Their product is 14. Find the numbers.
12. One number is two more than the other and their product is -1. Find the numbers.

See example 4–8 A–2.

13. The length of a rectangle is 2 meters less than twice the width. If the area is 24 square meters, what are the dimensions of the rectangle?
14. The area of a rectangle is 21 square feet. What are its dimensions if the length is 5 feet less than four times the width?
15. The area of a rectangle is numerically equal to twice the length. If the length is 3 feet more than the width, what are the dimensions of the rectangle?
16. The length of a rectangle is three less than twice the width. If the area is numerically five times the length, find the dimensions of the rectangle.
17. The height of a page of a book is 3 inches more than the width. If the area of the page is six less than ten times the width, find the width of the page.

18. The height of a page of a book is 4 inches more than the width. If there is a margin of 1 inch all around the printed matter of the page and the area of the printed matter is 32 square inches, what is the height of the page?

See example 4–8 A–3.

19. An object with initial velocity v undergoes an acceleration a for time t. The displacement s of the object for this time is given by the equation

$$s = vt + \frac{1}{2}at^2$$

a. Find t when $s = 8$, $v = 2$, $a = 2$
b. Find t when $s = 6$, $v = 3$, $a = 6$

20. The current in a circuit flows according to the equation $i = 12 - 12t^2$, where i is the current in amperes and t is the time in seconds. Find t when i is (a) 0 amperes, (b) 9 amperes.

21. The power output of a generator armature is given by $P_0 = E_gI - r_gI^2$.
a. Find I when $P_0 = 120$, $E_g = 22$, $r_g = 1$
b. Find I when $P_0 = 120$, $E_g = 16$, $r_g = \frac{1}{2}$

22. The output power P of a 100-volt electric generator is defined by

$$P = 100I - 5I^2$$

where I is in amperes. Find I when (a) $P = 480$, (b) $P = 375$

23. A ball rolls down a slope and travels a distance $d = 6t + \frac{t^2}{2}$ feet in t seconds. Find t when (a) $d = 14$ feet, (b) $d = 32$ feet.

24. Because of gravity, an object falls a distance s feet according to the formula $s = 16t^2$, where t seconds is the time it falls. How long will it take the object to fall (a) 256 feet; (b) 49 feet; (c) $2\frac{1}{4}$ feet; (d) 1,024 feet?

The formula $s = v_0t - 16t^2$ gives the height s in feet that an object will travel in t seconds if it is propelled directly upward at an *initial* velocity of v_0 feet per second. (Use this formula in exercises 25 through 28.)

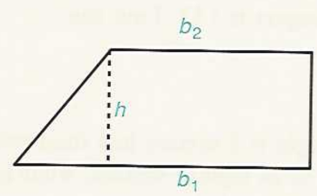
25. If an object is thrown upward at 96 feet per second, how long will it take the object to reach a height of 80 feet?

26. How long will it take before the object of exercise 25 hits the ground? (*Hint:* $s = 0$ when this happens.)

27. A projectile is fired upward with an initial velocity of 144 feet per second. How long will it take before the projectile strikes an object 288 feet directly overhead?

28. How long will it take before the projectile in exercise 27 falls back to the ground?

29. The formula for the area A of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$, where h is the altitude (height) of the trapezoid and the parallel bases are b_1 and b_2 (see diagram). If the area of the trapezoid is 63 square inches, base b_1 is 10 inches long, and the altitude h is 1 inch less than the length of b_2, find the length of h and b_2.

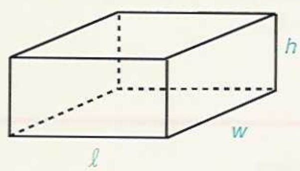


30. If the area of a trapezoid is 21 square feet, base b_2 is 5 feet long, and base b_1 is 6 feet longer than the altitude h, find the altitude of the trapezoid.

31. The altitude and base b_2 of a trapezoid have the same length. If the area of the trapezoid is 24 square meters and base b_1 is twice as long as base b_2, find the dimensions of the trapezoid.

32. One base of a trapezoid is three times the length of the other base. If the altitude is twice as long as the shorter base and the area is 36 square centimeters, find the dimensions of the trapezoid.

$\boxed{33.}$ The volume of a box (rectangular solid) is given by $V = \ell wh$, where ℓ is the length, w is the width, and h is the height of the box (see diagram). If the box is 4 feet tall, the length is 1 foot longer than the width, and the volume is 224 cubic feet, find the length and the width of the box.



34. A box is 9 inches long and has a volume of 162 cubic inches. If the width of the box is twice the height, find the width and the height of the box.

35. A storage room is three times as long as it is wide. If the room contains 756 cubic feet of space and has a ceiling 7 feet high, find the length and width of the room.

36. A cardboard box has a volume of 108 cubic inches. If the length of the box is 1 inch more than two times its width and the box has a height of 3 inches, find the length and the width of the box.

Review exercises

Find the solution set. See sections 2–6 and 4–7.

1. $3x - 2 = 0$
2. $4a - 5 = 0$
3. $x^2 - 9 = 0$
4. $x^2 + 6x + 8 = 0$

Evaluate the following expressions if $a = 4$, $b = -3$, $c = -6$, and $d = 5$. See section 2–2.

5. $ab - cd$
6. $b^2 - c^2$
7. $\dfrac{c - 2b}{d - a}$
8. $\dfrac{c + d}{3a + 4b}$

Chapter 4 lead-in problem

The formula $s = vt - 16t^2$ gives the height s in feet that an object will travel in t seconds if it is propelled directly upward at an initial velocity of v feet per second. If an object is thrown upward at 96 feet per second, how long will it take the object to reach a height of 144 feet?

Solution

$s = vt - 16t^2$	Original equation
$(144) = (96)t - 16t^2$	Substitute 144 for s and 96 for v
$16t^2 - 96t + 144 = 0$	Standard form
$16(t^2 - 6t + 9) = 0$	Common factor
$16(t - 3)(t - 3) = 0$	Factor the trinomial
$t - 3 = 0$	Set the repeated factor equal to zero
$t = 3$	Solve

The object will reach a height of 144 feet in 3 seconds.

Chapter 4 summary

1. **Common factors** are factors that appear in all the original terms.
2. A polynomial with integer coefficients is in **completely factored form** when
   a. the polynomial is written as a product of polynomials with integer coefficients;
   b. none of the polynomial factors, other than the monomial factor, can be factored further.
3. The **difference of two squares** factors as
   $$a^2 - b^2 = (a + b)(a - b)$$
4. The **trinomial** $ax^2 + bx + c$ will factor only if we can find a pair of integers, m and n, whose product is $a \cdot c$ and whose sum is b.
5. **Perfect square trinomials** factor as
   $$a^2 + 2ab + b^2 = (a + b)^2$$
   and
   $$a^2 - 2ab + b^2 = (a - b)^2$$
6. The **sum or difference of two cubes** factors as
   $$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
   and
   $$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
7. We try to factor **four-term polynomials** by grouping.
8. A **quadratic equation** is an equation that contains the second, but no higher, power of the variable.
9. The product of two factors $(x + p)(x + q) = 0$ only if $x + p = 0$ or $x + q = 0$

Chapter 4 error analysis

1. Factoring a common factor
   Example: $3x^2 - 6x - 3 = 3(x^2 - 2x)$
   Correct answer: $3(x^2 - 2x - 1)$
   What error was made? (*see page 158*)
2. Completely factoring a trinomial
   Example: $3x^2 - 6x + 3 = 3(x^2 - 2x + 1)$
   $= (x - 1)^2$
   Correct answer: $3(x - 1)^2$
   What error was made? (*see page 170*)
3. Factoring the sum of two squares
   Example: $4x^2 + 9y^2 = (2x + 3y)^2$
   Correct answer: $4x^2 + 9y^2$ is not factorable.
   What error was made? (*see page 177*)
4. Factoring the difference of two squares
   Example: $16x^2 - 4y^2 = (4x)^2 - (2y)^2$
   $= (4x + 2y)(4x - 2y)$
   Correct answer: $4(2x + y)(2x - y)$
   What error was made? (*see page 177*)
5. Factoring the sum of two cubes
   Example: $x^3 + y^3 = (x + y)(x^2 - 2xy + y^2)$
   Correct answer: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
   What error was made? (*see page 182*)
6. Order between real numbers
   Example: $-15 > 4$
   Correct answer: $-15 < 4$
   What error was made? (*see page 30*)
7. Combining using grouping symbols
   Example: $6 - (10 + 3) = 6 - 10 + 3 = -1$
   Correct answer: -7
   What error was made? (*see page 57*)
8. Multiplication of real numbers
   Example: $5\frac{3}{4} = 5 \cdot \frac{3}{4} = \frac{15}{4}$
   Correct answer: $\frac{23}{4}$
   What error was made? (*see page 46*)
9. Division by zero
   Example: $\frac{0}{0} = 1$
   Correct answer: $\frac{0}{0}$ is indeterminate.
   What error was made? (*see page 53*)
10. Exponents
    Example: $-2^2 = 4$
    Correct answer: -4
    What error was made? (*see page 57*)

Chapter 4 critical thinking

If n is an integer, for what values of n will $4n^2 + 10n + 4$ represent an even number?

Chapter 4 review

[4–1]

Supply the missing factors.

1. $3x + 9 = 3(\quad)$
2. $9x^2 - 18x = 9x(\quad)$
3. $-4y^3 + 8y^2 = -4y^2(\quad)$
4. $7a + 14b - 28c = 7(\quad)$
5. $-5a^2 - 15a + 30a^3 = -5a(\quad)$

Write in completely factored form.

6. $3a^2 - 3ab + 3b$
7. $x^2y + xyz + xy^2z$
8. $a^3b + a^3b^2$
9. $3R^3 - R^2 + 5R^4$
10. $4y^2 + 8y + 12y^3$
11. $x^4 + 3x^3 + 9x^2$
12. $16R^3S^2 - 12R^4S^3 + 24R^2S^2$
13. $10a^4b^3 + 15a^2b^2 - 20a^3b^2$
14. $2(a + b) + x(a + b)$
15. $y(x - 3z) + 4(x - 3z)$
16. $a(3R + 1) + b(3R + 1)$
17. $2a(x - 3y) - 3b(x - 3y)$
18. $6ax - 3ay - 2bx + by$
19. $4ax + 6by + 8ay + 3bx$
20. $ax + 3bx - 4a - 12b$
21. $ax^2 - 2bx^2 + 4a - 8b$

[4–2]

Write in completely factored form.

22. $x^2 - 9x + 14$
23. $2a^3 - 8a^2 - 10a$
24. $a^2 + 14a + 24$
25. $x^2 - 4x - 32$
26. $a^2 - 16a - 36$
27. $3x^2 - 9x - 30$
28. $x^3 - x^2 - 6x$
29. $x^3 - 4x^2 - 21x$
30. $a^2b^2 + ab - 6$
31. $a^2b^2 + 10ab + 24$
32. $a^2b^2 - 9ab + 18$
33. $a^2b^2 - 8ab - 20$

[4–3]

Write in completely factored form.

34. $4x^2 + 4x + 1$
35. $9r^2 - 36r + 36$
36. $4x^2 - 5x + 1$
37. $9a^2 + 9a - 10$
38. $8a^2 - 2a - 3$
39. $24x^2 + 22x + 3$
40. $8a^2 - 18a + 9$
41. $2a^2 + 15a + 18$

[4–4]

Write in completely factored form.

42. $4a^2 - 9$
43. $36b^2 - c^2$
44. $25 - a^2$
45. $16x^2 - 4y^2$
46. $9x^2 - y^4$
47. $x^4 - 16$
48. $y^4 - 81$
49. $b^2 + 12b + 36$
50. $c^2 - 10c + 25$
51. $4x^2 - 12x + 9$
52. $9x^2 - 12x + 4$

[4–5]

Write in completely factored form.

53. $R^3 + 8S^3$
54. $16x^3 - 54$
55. $27a^3 + 125b^3$
56. $x^3y^3 - 1$
57. $2x^9 + 250$
58. $64x^{12} - y^{15}$
59. $a^3b^6 + c^9$

[4–6]

Write in completely factored form.

60. $12x^4 - 3x^3$
61. $a^2 - 3a - 10$
62. $4a^2 - 21a + 5$
63. $9y^2 - 4$
64. $6ax + 9bx - 4a - 6b$
65. $b^2 - b - 20$
66. $9x^2 + 21x + 10$
67. $a^2 + 14a + 49$
68. $12x^5 - 3x^3$
69. $c^3 + 9c^2 + 20c$
70. $16a^2 - 8a + 1$
71. $b^4 - 1$

[4–7]

Determine the solution set.

72. $(x - 1)(x + 3) = 0$
73. $3x(x - 8) = 0$
74. $(5x + 1)(3x - 7) = 0$
75. $(7x - 1)(5x - 8) = 0$
76. $(4 - 3x)(9 - x) = 0$
77. $5x(x + 9)(3x + 12) = 0$
78. $(5x - 4)(5x + 4) = 0$
79. $(4x + 4)(5x - 15)(7x + 14) = 0$

Find the solution set of the following quadratic equations by factoring.

80. $4x^2 - 9x = 0$
81. $y^2 = 1$
82. $2a^2 = 128a$
83. $3x^2 - 75 = 0$
84. $x^2 - x - 30 = 0$
85. $2y = y^2 + 1$
86. $4a^2 + 13a + 3 = 0$
87. $5x^2 - 4 = 8x$
88. $x(x - 8) = -16$
89. $3(x^2 + 3) = 12x$
90. $8y^2 + y - 6 = -y$
91. $4a - 12 = 2a - 2a^2$

[4–8]

Solve the following word problems.

92. If the product of two consecutive integers is ten less than the square of the greater integer, determine the integers.

93. The length of a rectangular flower garden is 5 feet more than its width. If the area of the garden is 104 square feet, find the dimensions of the flower garden.

94. A farmer has some cattle to sell to a slaughterhouse. When the manager of the slaughterhouse asked how many cattle he had to sell, the farmer replied, "If you triple the square of the number of cattle you get 1,200." How many cattle did the farmer have to sell?

95. The height h in feet of a projectile launched vertically upward from the top of a 96-foot tall tower when time $t = 0$ is given by $h = 96 + 80t - 16t^2$. How long will it take the projectile to strike the ground?

Chapter 4 cumulative test

Perform the indicated operations and simplify.

[1–8] 1. $40 - 2 \cdot 8 \div 4 - 6 + 3$
[3–4] 2. $(2a^2b)^3$
[3–2] 3. $(a + 2b)^2$
[3–1] 4. $a^3 \cdot a^2 \cdot a$
[1–8] 5. $3[6 - 4(5 - 2) + 10]$
[3–1] 6. $(x^2y^3)(x^3y)$
[2–3] 7. $(3x - y) - (4y - 3x) - (x + 2y)$
[3–2] 8. $(3x - 2y)(3x + 2y)$
[3–3] 9. $\left(\dfrac{2a^2}{b}\right)^3$
[3–3] 10. $x^3 \cdot x^{-5}$
[2–3] 11. $3x - [x - y - (2x + 3y)]$
[3–4] 12. $(3x^{-3}y^2)^{-2}$
[2–2] 13. If $P(x) = x^2 - x - 6$, find (a) $P(-2)$, (b) $P(3)$, (c) $P(0)$.

Solve the inequalities and find the solution set for the equations.

[2–6] 14. $3(3x - 2) = 4x + 3$
[2–9] 15. $\dfrac{3}{4}x + 2 < \dfrac{1}{2}x + 5$
[2–6] 16. $2(2x - 1) = 6(x - 2) - 4$
[2–9] 17. $2x + 7 \geq 13$
[4–7] 18. $x^2 - 9 = 0$
[4–7] 19. $x^2 - 7x + 10 = 0$
[2–9] 20. $3(2x + 1) < 4x + 8$
[2–9] 21. $-2 \leq 3x - 8 \leq 6$

Solve for the specified variable.

[2–7] **22.** $3x - y = x + 5y$ for x

[2–7] **23.** $2x + 5 - 3y = 5x + 3 - 8y$ for x

Write in completely factored form.

[4–1] **24.** $2ab - 4a^2b^2 - 8a^3b^5$

[4–4] **25.** $4a^2 + 12a + 9$

[4–4] **26.** $25c^2 - 9d^2$

[4–3] **27.** $4a^2 - 4a - 15$

[4–2] **28.** $x^2 + 9x + 18$

Set up an equation and solve for the unknown(s).

[2–8] **29.** One number is eleven more than twice a second number. If their sum is 53, what are the numbers?

[4–8] **30.** The product of two consecutive positive odd integers is 143. Find the integers.

[2–8] **31.** Terry has \$15,000. He invests part of this money at 8% and the rest at 6%, and his income for one year from these investments totals \$1,100. How much was invested at each rate?

[4–8] **32.** The area of a rectangle is six more than ten times the width. If the length is 5 meters longer than the width, what are the dimensions of the rectangle?

Exercise 3–5

Answers to odd-numbered problems

1. 2.55×10^2 **3.** 1.2345×10^4 **5.** 1.55×10^5
7. 8.55076×10^2 **9.** 1.0076×10^6 **11.** 1.2×10^{-4}
13. 8.1×10^{-6} **15.** 7×10^{-4} **17.** 9.4×10^{-11}
19. -4.5×10^3 **21.** -5.85×10^6 **23.** -4.578×10
25. -2.985×10^{-8} **27.** 49,900,000 **29.** 7.23 **31.** 0.0042
33. 0.00000147 **35.** 0.000789 **37.** -0.00000000482
39. -0.00000492 **41.** 1×10^{-9} **43.** 2×10^{12}
45. 35,600,000 **47.** 0.000 000 000 000 000 000 000 093
49. 140,000 **51.** 1.22304×10^{14} **53.** 3.63226×10^{-1}
55. 1.76979×10^{-7} **57.** 4.84481×10^8 **59.** 1.4×10^3
61. 4.6×10^3

Solution to trial exercise problem

58. $(177{,}000) \div (0.15) = \dfrac{1.77 \times 10^5}{1.5 \times 10^{-1}}$
$= \dfrac{1.77}{1.5} \times 10^{5-(-1)} = \dfrac{1.77}{1.5} \times 10^6 = 1.18 \times 10^6$

Review exercises

1. $5x$ **2.** $3a^2$ **3.** $12ab$ **4.** $x^3 + 2x^2$ **5.** $6a^2 - 15a$
6. $3x^3y + 2x^2y^2 - 7x^2y$

Chapter 3 review

1. a^{12} **2.** a^{14} **3.** $4^5 = 1{,}024$ **4.** x^4y^4 **5.** a^{15} **6.** $20a^4b^5$
7. $6x^3y^7$ **8.** $-15x^5$ **9.** $6a^3b^5$ **10.** $10x^5y^5$ **11.** $-6a^6b^{10}$
12. $15x^2 - 10xy$ **13.** $-6a^4b + 9a^3b^2 - 12a^2b^3$
14. $30x^2y^2 - 10xy^3$ **15.** $x^2 - x - 12$ **16.** $x^2 + 10x + 25$
17. $a^2 - 49$ **18.** $15x^2 + 7xy - 2y^2$ **19.** $x^3 + x^2y - 5xy^2 - 2y^3$
20. $3a^3 - 4a^2b - 5ab^2 + 2b^3$ **21.** $8a^3 + 12a^2b + 6ab^2 + b^3$
22. $\dfrac{1}{b^2}$ **23.** $\dfrac{5}{a^2}$ **24.** a^4 **25.** $\dfrac{1}{x^3}$ **26.** 1 **27.** $\dfrac{5}{x^3y^2}$ **28.** a^3
29. $\dfrac{a^5}{b^5}$ **30.** $\dfrac{4y^2z^2}{x^2}$ **31.** $\dfrac{b^3}{4a^3}$ **32.** $\dfrac{a^9}{b^3}$ **33.** $8a^6b^9$
34. $3^{12}x^{16}y^{20} = 531{,}441x^{16}y^{20}$ **35.** $\dfrac{y^6}{4x^2}$ **36.** $\dfrac{xy^3z^3}{2}$ **37.** $\dfrac{2a^2c^3}{b^6}$
38. $8x^{26}y^{25}$ **39.** $72a^{14}b^3$ **40.** $\dfrac{a^9b^3}{c^{12}}$ **41.** $\dfrac{9a^6b^4}{c^{10}}$ **42.** $\dfrac{32x^5y^{20}}{z^{30}}$
43. 1.84×10^3 **44.** 1.57×10^{-3} **45.** 1.07×10^8
46. 8.49×10^{11} **47.** -3.75×10 **48.** -5.43×10^{-3}
49. 504,000 **50.** 0.00639 **51.** -596 **52.** -0.00886
53. 0.000000735 **54.** 812,000,000 **55.** 2.67672×10^5
56. 1.54818×10^{-8} **57.** 7.2×10^{-3} **58.** 1.25×10^{-7}

Chapter 3 cumulative test

1. false **2.** false **3.** true **4.** -2 **5.** undefined **6.** 8
7. 0 **8.** 16 **9.** 38 **10.** $x^2y + 2xy^2 - 3x^2y^2$ **11.** -40
12. 0 **13.** $9x^2 - 6xy + y^2$ **14.** 0 **15.** $4a^4b^{10}$ **16.** -25
17. 0 **18.** $9x^2 - 4y^2$ **19.** $6x^3y^5$ **20.** 26 **21.** undefined
22. $4a + 2b$ **23.** $x^3 - 2x - 1$ **24.** x^2 **25.** a^4 **26.** $\{7\}$
27. $\left\{\dfrac{5}{3}\right\}$ **28.** $\left\{-\dfrac{19}{4}\right\}$ **29.** $x > -\dfrac{1}{3}$ **30.** $x > \dfrac{1}{6}$
31. $\left\{-\dfrac{3}{2}\right\}$ **32.** $-8 \le x \le -1$ **33.** 97 **34.** 72
35. \$18,500 at 8%; \$11,500 at 7%

Chapter 4

Exercise 4–1

Answers to odd-numbered problems

1. $2(y + 3)$ **3.** $4(x^2 + 2y)$ **5.** $3(x^2y + 5z)$
7. $7(a - 2b + 3c)$ **9.** $3(5xy - 6z + x^2)$
11. $7(6xy - 3y^2 + 1)$ **13.** $2(4x - 5y + 6z - 9w)$
15. $5ab(4a - 12 + 9b)$ **17.** $3xy(x + 2)$ **19.** $2R^2(R^2 - 3)$
21. $x(2x^2 - x + 1)$ **23.** $3ab(5 + 6b - a)$ **25.** $xy(y + z + 1)$
27. $2L(L^2 - 9 + L)$ **29.** $5p(p + 2 + 3p^2)$ **31.** $-3(2x + 3)$
33. $2(3x - 4z - 6w)$ **35.** $-3(4L - 5W + 2H)$
37. $-x(1 - x + x^2)$ **39.** $-xyz(1 - x + y - z)$
41. $-5ab(2ab - 3 + 4a^2b^2)$ **43.** $(a + b)(x + y)$
45. $5(2a + b)(3x + 2y)$ **47.** $3(a + 4b)(x + 2y)$
49. $(b + 6)(8a - 1)$ **51.** $(a + b)(c + d)$
53. $(2a + b)(3x - 2y)$ **55.** $(2x + y)(2a - b)$
57. $(5x - 3y)(4x + z)$ **59.** $(a + 3b)(4x - 3y)$
61. $(2c - y)(a + 3b)$ **63.** $(c + 4y)(2a + 3b)$
65. $(3x + y)(2a + b)$ **67.** $(x - 2d)(3a + b)$
69. $(a + 5)(2a^2 + 3)$ **71.** $(4a^2 + 3)(2a - 1)$
73. $\pi r(s + r)$ **75.** $\dfrac{wx}{48EI}(2x^3 - 3\ell x^2 - \ell^3)$

Solutions to trial exercise problems

11. $42xy - 21y^2 + 7 = 7 \cdot 6xy - 7 \cdot 3y^2 + 7 \cdot 1$
$= 7(6xy - 3y^2 + 1)$ **25.** $xy^2 + xyz + xy = xy \cdot y + xy \cdot z$
$+ xy \cdot 1 = xy(y + z + 1)$ **33.** $6x - 8z - 12w = 2(\quad)$;
$2 \cdot 3x - 2 \cdot 4z - 2 \cdot 6w = 2(3x - 4z - 6w)$
34. $-4a^3 - 36ab + 16ab^2 - 24b^3 = -4(\quad)$;
$(-4)a^3 + (-4)(9ab) + (-4)(-4ab^2) + (-4)(6b^3)$
$= -4(a^3 + 9ab - 4ab^2 + 6b^3)$ **36.** $-3a + a^3b = -a(\quad)$;
$(-a) \cdot 3 + (-a)(-a^2b) = -a(3 - a^2b)$ **45.** $15x(2a + b)$
$+ 10y(2a + b) = 5(2a + b) \cdot 3x + 5(2a + b) \cdot 2y$
$= 5(2a + b)(3x + 2y)$ **53.** $6ax - 2by + 3bx - 4ay$
$= 6ax + 3bx - 4ay - 2by = (6ax + 3bx) + (-4ay - 2by)$
$= 3x(2a + b) - 2y(2a + b) = (2a + b)(3x - 2y)$
75. $Y = \dfrac{2wx^4}{48EI} - \dfrac{3\ell wx^3}{48EI} - \dfrac{\ell^3 wx}{48EI} = \dfrac{wx}{48EI} \cdot 2x^3 - \dfrac{wx}{48EI} - 3\ell x^2$
$- \dfrac{wx}{48EI} \cdot \ell^3 = \dfrac{wx}{48EI}(2x^3 - 3\ell x^2 - \ell^3)$

Review exercises

1. 4, 5 **2.** 3, 4 **3.** -8, 2 **4.** 8, -2 **5.** 8, 2 **6.** -8, -2
7. 6, 6 **8.** 11, 1

Exercise 4–2

Answers to odd-numbered problems

1. $(a + 6)(a + 3)$ **3.** $(x + 12)(x - 1)$ **5.** $(y + 15)(y - 2)$
7. $(x - 12)(x - 2)$ **9.** $(a + 8)(a - 3)$ **11.** $(x + 6)(x + 2)$
13. $(a - 6)(a + 4)$ **15.** $2(x + 5)(x - 2)$ **17.** $3(x - 8)(x + 2)$
19. will not factor, prime polynomial **21.** $(y + 15)(y + 2)$
23. $4(x + 2)(x - 3)$ **25.** $5(a - 5)(a + 2)$
27. $(xy - 6)(xy + 3)$ **29.** $(xy + 12)(xy + 1)$
31. $3(xy + 3)(xy - 4)$ **33.** $(x + 2y)(x + y)$
35. $(a - 3b)(a + b)$ **37.** $(a - 3b)(a + 2b)$
39. $(x + 3y)(x - 5y)$

Solutions to trial exercise problems

17. $3x^2 - 18x - 48 \quad = 3(x^2 - 6x - 16)$
m and n are -8 and 2. $= 3(x - 8)(x + 2)$
26. $x^2y^2 - 4xy - 21 \quad = (xy)^2 - 4(xy) - 21$
m and n are -7 and 3. $= (xy - 7)(xy + 3)$
33. We need to find m and n that add to $3y$ and multiply to $2y^2$. The values are $2y$ and y. The factorization is $x^2 + 3xy + 2y^2 = (x + 2y)(x + y)$.

Review exercises

1. $x^2(a + b + c)$ **2.** $3x(x^2 + 4x - 2)$ **3.** $(3x + 5)(2x + 1)$
4. $(2x + 3)(3x - 2)$ **5.** $(4x + 1)(5x + 1)$
6. $(6x - 1)(2x + 3)$ **7.** $(x - 2)(3x - 5)$
8. $(7x - 3)(x - 9)$

Exercise 4–3

Answers to odd-numbered problems

1. $(2x - 3)(x + 2)$ **3.** $(2x + 1)(x + 1)$ **5.** $(2R - 3)(R - 2)$
7. $(5x + 3)(x - 2)$ **9.** $(3x - 1)(3x - 1) = (3x - 1)^2$
11. will not factor, prime polynomial **13.** $(2x + 3)(3x + 2)$
15. $(2x + 7)(2x + 3)$ **17.** will not factor, prime polynomial
19. $(3y - 8)(3y + 1)$ **21.** $(5x + 6)(2x - 1)$
23. $(2x - 5)(x - 2)$ **25.** $2(2x + 3)(x + 2)$
27. $2(2x + 3)(x + 1)$ **29.** $(2x + 3)(3x - 1)$
31. $2(x + 5)(x - 2)$ **33.** $(3x - 2)(2x + 3)$
35. $(2x + 3)(2x + 3) = (2x + 3)^2$ **37.** $(7x - 1)(x - 5)$
39. $(3P + 1)(5P - 1)$ **41.** $2x(x - 5)(x + 2)$
43. $a(2a + 1)(a + 7)$ **45.** $(2x - 5)(4x + 3)$
47. $-16(t - 1)(t - 1) = -16(t - 1)^2$

Solutions to trial exercise problems

25. $4x^2 + 14x + 12 \quad = 2(2x^2 + 7x + 6)$
m and n are 3 and 4. $= 2[(2x^2 + 3x) + (4x + 6)]$
$= 2[x(2x + 3) + 2(2x + 3)]$
$= 2(2x + 3)(x + 2)$
41. $2x^3 - 6x^2 - 20x \quad = 2x(x^2 - 3x - 10)$
m and n are -5 and 2. $= 2x(x - 5)(x + 2)$

Review exercises

1. $x^2 - y^2$ **2.** $9a^2 - 4b^2$ **3.** $x^2 - 2xy + y^2$ **4.** $25a^2 - 16b^2$
5. $4a^2 + 4ab + b^2$ **6.** $16x^2 - 8xy + y^2$ **7.** $x^4 - 1$
8. $a^4 - 16$

Exercise 4–4

Answers to odd-numbered problems

1. $(6)^2$ **3.** $(c)^2$ **5.** $(4x)^2$ **7.** $(2z^2)^2$ **9.** $(x + 1)(x - 1)$
11. $(a + 2)(a - 2)$ **13.** $(3 + E)(3 - E)$ **15.** $(1 + k)(1 - k)$
17. $(3b + 4)(3b - 4)$ **19.** $(b + 6c)(b - 6c)$
21. $(2a + 5b)(2a - 5b)$ **23.** $(5p + 9)(5p - 9)$
25. $8(x + 2y)(x - 2y)$ **27.** $5(r + 5s)(r - 5s)$
29. $2(5 + x)(5 - x)$ **31.** $(rs + 5t)(rs - 5t)$
33. $(x^2 - 3)(x^2 + 3)$ **35.** $(r^2 + 9)(r + 3)(r - 3)$
37. $(7x + 8y^2)(7x - 8y^2)$ **39.** $2(7xy + 5pc)(7xy - 5pc)$
41. $(c - 7)^2$ **43.** $(a + 3)^2$ **45.** $(y - 3)^2$ **47.** $(2a - 3b)^2$
49. $(3c - 2d)^2$ **51.** $\frac{V}{8I}(h + 2v_1)(h - 2v_1)$

Solutions to trial exercise problems

17. $9b^2 - 16 = (3b)^2 - (4)^2 = (3b + 4)(3b - 4)$
25. $8x^2 - 32y^2 = 8(x^2 - 4y^2) = 8[(x)^2 - (2y)^2]$
$= 8(x + 2y)(x - 2y)$ **34.** $x^4 - 1 = (x^2)^2 - (1)^2$
$= (x^2 + 1)(x^2 - 1) = (x^2 + 1)[(x)^2 - (1)^2]$
$= (x^2 + 1)(x + 1)(x - 1)$ **41.** $c^2 - 14c + 49$
$= (c)^2 - 2(c)(7) + (7)^2 = (c - 7)^2$

Review exercises

1. $(x + 6)(x + 2)$ **2.** $(7a + 9)(7a - 9)$ **3.** $(x - 4y)(3a + b)$
4. $2x(x + 3)(x + 4)$ **5.** $(2a + 3)(5a + 3)$ **6.** $(2a - 5)^2$
7. $(xy + 5)(xy + 3)$ **8.** $(x + 2y)^2$

Exercise 4–5

Answers to odd-numbered problems

1. 4^3 **3.** 5^3 **5.** $(3x)^3$ **7.** $(a^2)^3$ **9.** $(2b^5)^3$
11. $(r + s)(r^2 - rs + s^2)$ **13.** $(2x + y)(4x^2 - 2xy + y^2)$
15. $(h - k)(h^2 + hk + k^2)$ **17.** $(a - 2)(a^2 + 2a + 4)$
19. $(x - 2y)(x^2 + 2xy + 4y^2)$ **21.** $(4x - y)(16x^2 + 4xy + y^2)$
23. $(3x - 2y)(9x^2 + 6xy + 4y^2)$
25. $(2a + 3b)(4a^2 - 6ab + 9b^2)$
27. $2(a + 2)(a^2 - 2a + 4)$ **29.** $2(x - 2)(x^2 + 2x + 4)$
31. $x^2(x + 3y)(x^2 - 3xy + 9y^2)$ **33.** $(x^2 + y)(x^4 - x^2y + y^2)$
35. $(a^3 - b)(a^6 + a^3b + b^2)$ **37.** $(x^4 - 3)(x^8 + 3x^4 + 9)$
39. $a^2(2b - a)(4b^2 + 2ab + a^2)$ **41.** $2(3r + s)(9r^2 - 3rs + s^2)$
43. $(xy - z)(x^2y^2 + xyz + z^2)$
45. $(a^5b^2 - 2c^3)(a^{10}b^4 + 2a^5b^2c^3 + 4c^6)$
47. $(ab + 2)(a^2b^2 - 2ab + 4)$
49. $(x^3y^4 + z^5)(x^6y^8 - x^3y^4z^5 + z^{10})$

Solutions to trial exercise problems

27. $2a^3 + 16 = 2(a^3 + 8)$
$= 2[(a)^3 + (2)^3]$
Then $2(\ \ +\ \ )[(\ \ )^2 - (\ \ )(\ \ ) + (\ \ )^2]$
and $= 2(a + 2)[(a)^2 - (a)(2) + (2)^2]$
$= 2(a + 2)(a^2 - 2a + 4)$
44. $x^3y^9 - 1 = (xy^3)^3 - (1)^3$
Then $(\ \ -\ \ )[(\ \ )^2 + (\ \ )(\ \ ) + (\ \ )^2]$
and $(xy^3 - 1)[(xy^3)^2 + (xy^3)(1) + (1)^2]$
$= (xy^3 - 1)(x^2y^6 + xy^3 + 1)$

Review exercises

1. $(a - 2)(a - 5)$ **2.** $(3a + b)(2x - y)$ **3.** $(x + 2y)^2$
4. $(3a + b)(2a - b)$ **5.** $5a(a - 3)(a - 5)$
6. $(2x + 3)(3x - 4)$

Exercise 4–6

Answers to odd-numbered problems

1. $(n + 7)(n - 7)$ **3.** $(7b + 1)(b + 5)$ **5.** $(xy + 4)(xy - 2)$
7. $(6 + y)(6 - y)$ **9.** $10(a - b)^2$ **11.** $4(a + 2b)(a - 2b)$
13. $(3a - b)(x + 2y)$ **15.** $(3x + 5)(2x - 1)$
17. $(2m - n)(3a + 2b)$ **19.** $(7b - 5)(b + 3)$
21. $(4x - 3)(x + 5)$ **23.** $6(x^2 - 4xy - 8y^2)$
25. $3xy(x + 5y)(m - 4n)$ **27.** $(3a - 5b)^2$
29. $3a(a^2 + 4)(a + 2)(a - 2)$ **31.** $3ab^3(a + b)^2$
33. $(3b + 7)(b - 13)$ **35.** $(3x + 2y)(a + 2b)$
37. $(6x - 1)(x + 2)$ **39.** $3x^2(x + 4)(x - 4)$
41. $(y + 3z)(y^2 - 3yz + 9z^2)$ **43.** $(x - y^3)(x^2 + xy^3 + y^6)$

Solutions to trial exercise problems

25. $3x^2y(m-4n) + 15xy^2(m-4n) = 3xy(m-4n)(x+5y)$ common factor of $3xy(m-4n)$ **27.** $9a^2 - 30ab + 25b^2 = (3a)^2 - 2(3a)(5b) + (5b)^2 = (3a-5b)^2$ **29.** $3a^5 - 48a = 3a(a^4 - 16) = 3a[(a^2)^2 - (4)^2] = 3a(a^2+4)(a^2-4) = 3a(a^2+4)(a+2)(a-2)$

Review exercises

1. $\{-3\}$ **2.** $\{3\}$ **3.** $\{6\}$ **4.** $\left\{-\frac{3}{5}\right\}$

5. $\left\{-\frac{2}{3}\right\}$ **6.** $\left\{-\frac{1}{3}\right\}$ **7.** $\left\{\frac{1}{4}\right\}$ **8.** $\{0\}$

Exercise 4–7

Answers to odd-numbered problems

1. $\{-5,5\}$ **3.** $\{0,-6\}$ **5.** $\{0,7\}$ **7.** $\left\{3,-\frac{3}{2}\right\}$ **9.** $\left\{-\frac{1}{2},\frac{2}{3}\right\}$

11. $\left\{-\frac{1}{5},\frac{1}{5}\right\}$ **13.** $\{7,8\}$ **15.** $\left\{\frac{4}{3},\frac{8}{5}\right\}$ **17.** $\{-5,5\}$

19. $\left\{-1,0,\frac{3}{2}\right\}$ **21.** $\left\{-\frac{3}{5},\frac{1}{4},10\right\}$ **23.** $\{0,-4\}$ **25.** $\left\{0,\frac{5}{3}\right\}$

27. $\{0,-3\}$ **29.** $\{-3,3\}$ **31.** $\left\{0,-\frac{3}{2}\right\}$ **33.** $\{-5,5\}$

35. $\{-3,3\}$ **37.** $\{-4,4\}$ **39.** $\{-2,2\}$ **41.** $\left\{-\frac{3}{2},\frac{3}{2}\right\}$

43. $\{-8,2\}$ **45.** $\{-7\}$ **47.** $\{-7,2\}$ **49.** $\{-1,-2\}$

51. $\{-1,12\}$ **53.** $\{-4,8\}$ **55.** $\left\{-1,\frac{9}{2}\right\}$ **57.** $\left\{\frac{1}{3},\frac{1}{2}\right\}$

59. $\left\{\frac{4}{3},-\frac{3}{2}\right\}$ **61.** $\left\{-\frac{1}{2},\frac{3}{2}\right\}$ **63.** $\left\{-\frac{1}{9},1\right\}$

65. $\left\{-\frac{4}{3},\frac{5}{2}\right\}$ **67.** $\{-1\}$ **69.** $\left\{-\frac{5}{3}\right\}$ **71.** $\left\{-\frac{7}{3},4\right\}$

73. $\left\{\frac{3}{2},-3\right\}$ **75.** $\{-2,-1\}$ **77.** $\{-2,-1\}$ **79.** $\{-2,9\}$

81. $\left\{\frac{2}{3}\right\}$ **83.** $\{-1,5\}$

Solutions to trial exercise problems

1.
$$\begin{aligned}
(x+5)(x-5) &= 0\\
x+5 = 0 \text{ or } x-5 &= 0\\
x = -5 \qquad x &= 5\\
&\{-5,5\}
\end{aligned}$$

7.
$$\begin{aligned}
(3x-9)(2x+3) &= 0\\
3x-9 = 0 \text{ or } 2x+3 &= 0\\
3x = 9 \qquad 2x &= -3\\
x = 3 \qquad x &= -\frac{3}{2}\\
&\left\{3,-\frac{3}{2}\right\}
\end{aligned}$$

15.
$$\begin{aligned}
(4-3u)(8-5u) &= 0\\
4-3u = 0 \text{ or } 8-5u &= 0\\
4 = 3u \qquad 8 &= 5u\\
\frac{4}{3} = u \qquad \frac{8}{5} &= u\\
&\left\{\frac{4}{3},\frac{8}{5}\right\}
\end{aligned}$$

31.
$$\begin{aligned}
10a^2 &= -15a\\
10a^2 + 15a &= 0\\
5a(2a+3) &= 0\\
5a = 0 \text{ or } 2a+3 &= 0\\
a = 0 \qquad 2a &= -3\\
a &= -\frac{3}{2}\\
&\left\{0,-\frac{3}{2}\right\}
\end{aligned}$$

40.
$$\begin{aligned}
5y^2 - 45 &= 0\\
5(y^2-9) &= 0\\
5(y+3)(y-3) &= 0\\
y+3 = 0 \text{ or } y-3 &= 0\\
y = -3 \qquad y &= 3\\
&\{3,-3\}
\end{aligned}$$

47.
$$\begin{aligned}
b^2 + 5b - 14 &= 0\\
(b+7)(b-2) &= 0\\
b+7 = 0 \text{ or } b-2 &= 0\\
b = -7 \qquad b &= 2\\
&\{-7,2\}
\end{aligned}$$

52.
$$\begin{aligned}
x^2 - 14x &= 15\\
x^2 - 14x - 15 &= 0\\
(x-15)(x+1) &= 0\\
x-15 = 0 \text{ or } x+1 &= 0\\
x = 15 \qquad x &= -1\\
&\{15,-1\}
\end{aligned}$$

59.
$$\begin{aligned}
6x^2 + x - 12 &= 0\\
(3x-4)(2x+3) &= 0\\
3x-4 = 0 \text{ or } 2x+3 &= 0\\
3x = 4 \qquad 2x &= -3\\
x = \frac{4}{3} \qquad x &= \frac{-3}{2}\\
&\left\{\frac{4}{3},\frac{-3}{2}\right\}
\end{aligned}$$

64.
$$\begin{aligned}
-6x &= -3x^2 - 3\\
3x^2 - 6x + 3 &= 0\\
3(x^2 - 2x + 1) &= 0\\
3(x-1)^2 &= 0\\
x - 1 &= 0\\
x &= 1\\
&\{1\}
\end{aligned}$$

71.
$$\begin{aligned}
3x^2 - 4x - 28 &= x\\
3x^2 - 5x - 28 &= 0\\
(3x+7)(x-4) &= 0\\
3x+7 = 0 \text{ or } x-4 &= 0\\
3x = -7 \qquad x &= 4\\
x = \frac{-7}{3} \qquad x &= 4\\
&\left\{\frac{-7}{3},4\right\}
\end{aligned}$$

76.
$$\begin{aligned}
3x(3x+2) &= 24\\
9x^2 + 6x &= 24\\
9x^2 + 6x - 24 &= 0\\
3(3x^2 + 2x - 8) &= 0\\
3(3x-4)(x+2) &= 0\\
3x-4 = 0 \text{ or } x+2 &= 0\\
3x = 4 \qquad x &= -2\\
x = \frac{4}{3} \qquad x &= -2\\
&\left\{\frac{4}{3},-2\right\}
\end{aligned}$$

84.
$$\begin{aligned} x(x+7) &= 36-2x \\ x^2+7x &= 36-2x \\ x^2+9x-36 &= 0 \\ (x+12)(x-3) &= 0 \\ x+12=0 &\text{ or } x-3=0 \\ x=-12 & \quad x=3 \\ \{-12,3\} \end{aligned}$$

Review exercises

1. (let x = the number) $x + 7$ **2.** (let x = the number) $x - 11$

3. $6(x^2 + x)$ **4.** $\frac{x^2 + 2x}{8}$ **5.** 13 **6.** 6, 32 **7.** 22, 24, 26

Exercise 4–8

Answers to odd-numbered problems

1. 11 and 13; -13 and -11 **3.** 16 and 18; -18 and -16 **5.** 7 and 13; -13 and -7 **7.** 4 and 6; -2 and 0 **9.** -4 and -9 **11.** 2 and 7; -6 and $-\frac{7}{3}$ **13.** $w = 4$ meters; $\ell = 6$ meters **15.** $w = 2$ feet; $\ell = 5$ feet **17.** $w = 1$ inch or $w = 6$ inches **19. a.** $t = 2$ **b.** $t = 1$ **21. a.** $I = 12$ or $I = 10$ **b.** $I = 12$ or $I = 20$ **23. a.** $t = 2$ **b.** $t = 4$ **25.** $t = 1$ second (on way up); $t = 5$ seconds (on way down) **27.** $t = 3$ seconds **29.** $b_2 = 8$ inches; $h = 7$ inches **31.** $b_2 = 4$ inches; $b_1 = 8$ inches; $h = 4$ inches **33.** $w = 7$ feet; $\ell = 8$ feet **35.** $w = 6$ feet; $\ell = 18$ feet

Solutions to trial exercise problems

7. Let x = the lesser integer.
Let $x + 2$ = the next consecutive even integer.
Then $x(x + 2) = 2[x + (x + 2)] + 4$
$$\begin{aligned} x^2+2x &= 2(2x+2)+4 \\ x^2+2x &= 4x+4+4 \\ x^2+2x &= 4x+8 \\ x^2-2x-8 &= 0 \\ (x-4)(x+2) &= 0 \\ x-4=0 &\text{ or } x+2=0 \\ x=4 & \quad x=-2 \\ x+2=6 & \quad x+2=0 \end{aligned}$$
The consecutive even integers are 4 and 6 or -2 and 0.

10. Let x = one integer.
Then $-3 - x$ = the other integer.
$$\begin{aligned} x(-3-x) &= -70 \\ -3x-x^2 &= -70 \\ 0 = x^2+3x-70 &= (x+10)(x-7) \\ x+10=0 &\text{ or } x-7=0 \\ x=-10 & \quad x=7 \end{aligned}$$
$-3 - x = -3 - (-10) = 7$ or $-3 - x = -3 - 7 = -10$
The integers are -10 and 7.

15. Let w = the width. Let $\ell = w + 3$. From "the area of a rectangle is numerically equal to twice the length," we get
$$\begin{aligned} w(w+3) &= 2(w+3) \\ w^2+3w &= 2w+6 \\ w^2+w-6 &= 0 \\ (w+3)(w-2) &= 0 \\ w+3=0 &\text{ or } w-2=0 \\ w=-3 & \quad w=2 \end{aligned}$$
(*Note:* A geometric figure cannot have a negative width, so we ignore $w = -3$.)
The width is 2 feet and the length is $w + 3 = 5$ feet.

22a. $P = 100I - 5I^2$, when $P = 480$.
$$\begin{aligned} 480 &= 100I - 5I^2 \\ 5I^2 - 100I + 480 &= 0 \\ 5(I^2 - 20I + 96) &= 0 \\ 5(I-8)(I-12) &= 0 \\ I-8=0 &\text{ or } I-12=0 \\ I=8 & \quad I=12 \end{aligned}$$
Therefore $I = 8$ amperes or $I = 12$ amperes.

25. $s = v_0t - 16t^2$, given $s = 80$ and $v_0 = 96$
$$\begin{aligned} 80 &= 96t - 16t^2 \\ 16t^2 - 96t + 80 &= 0 \\ 16(t^2 - 6t + 5) &= 0 \\ 16(t-5)(t-1) &= 0 \\ t-5=0 &\text{ or } t=1 \\ t=5 & \quad t=1 \end{aligned}$$
So $t = 1$ second on the way up,
and $t = 5$ seconds on the way down.

30. Given $A = \frac{1}{2}h(b_1 + b_2)$, $A = 21$ square feet, $b_2 = 5$ feet, and using "b_1 is 6 feet longer than the altitude h," we have $b_1 = h + 6$.
Substitute $21 = \frac{1}{2}h[(h + 6) + 5]$
$$21 = \frac{1}{2}h(h + 11)$$
Multiply both members by 2 to get $42 = h(h + 11)$
Then $42 = h^2 + 11h$
Add -42 to both members to get $0 = h^2 + 11h - 42$
So $h^2 + 11h - 42 = 0$
Factor in the left member: $(h + 14)(h - 3) = 0$
Then $h + 14 = 0$ or $h - 3 = 0$
So $h = -14$ or $h = 3$.
A trapezoid cannot have a negative altitude, so we ignore $h = -14$. The altitude of the trapezoid is 3 feet.

33. Given $V = \ell wh$, $V = 224$ cubic feet and $h = 4$ feet, from "the length is 1 foot longer than the width," $\ell = w + 1$.
Substituting, $224 = w(w + 1) \cdot 4$
Multiplying in the right member, $224 = 4w^2 + 4w$
Add -224 to both members and interchange members.
$$0 = 4w^2 + 4w - 224$$
$4w^2 + 4w - 224 = 0$
Factor 4 from each term to get $4(w^2 + w - 56) = 0$
Then $4(w + 8)(w - 7) = 0$ and $w + 8 = 0$ or $w - 7 = 0$
So $w = -8$ or $w = 7$.
The width of a box cannot be negative, so we discard $w = -8$.
The width is then 7 feet and the length is $w + 1 = 8$ feet.

Review exercises

1. $\left\{\frac{2}{3}\right\}$ **2.** $\left\{\frac{5}{4}\right\}$ **3.** $\{-3,3\}$ **4.** $\{-4,-2\}$ **5.** 18 **6.** -27 **7.** 0 **8.** undefined

Chapter 4 review

1. $3(x + 3)$ **2.** $9x(x - 2)$ **3.** $-4y^2(y - 2)$ **4.** $7(a + 2b - 4c)$ **5.** $-5a(a + 3 - 6a^2)$ **6.** $3(a^2 - ab + b)$ **7.** $xy(x + z + yz)$ **8.** $a^3b(1 + b)$ **9.** $R^2(3R - 1 + 5R^2)$ **10.** $4y(y + 2 + 3y^2)$ **11.** $x^2(x^2 + 3x + 9)$ **12.** $4R^2S^2(4R - 3R^2S + 6)$ **13.** $5a^2b^2(2a^2b + 3 - 4a)$ **14.** $(a + b)(2 + x)$ **15.** $(x - 3z)(y + 4)$

16. $(3R + 1)(a + b)$ **17.** $(x - 3y)(2a - 3b)$
18. $(3a - b)(2x - y)$ **19.** $(x + 2y)(4a + 3b)$
20. $(a + 3b)(x - 4)$ **21.** $(x^2 + 4)(a - 2b)$
22. $(x - 7)(x - 2)$ **23.** $2a(a - 5)(a + 1)$
24. $(a + 12)(a + 2)$ **25.** $(x - 8)(x + 4)$
26. $(a - 18)(a + 2)$ **27.** $3(x - 5)(x + 2)$
28. $x(x - 3)(x + 2)$ **29.** $x(x - 7)(x + 3)$
30. $(ab + 3)(ab - 2)$ **31.** $(ab + 6)(ab + 4)$
32. $(ab - 6)(ab - 3)$ **33.** $(ab - 10)(ab + 2)$
34. $(2x + 1)(2x + 1)$ or $(2x + 1)^2$ **35.** $9(r - 2)(r - 2)$ or $9(r - 2)^2$ **36.** $(4x - 1)(x - 1)$ **37.** $(3a + 5)(3a - 2)$
38. $(4a - 3)(2a + 1)$ **39.** $(6x + 1)(4x + 3)$
40. $(4a - 3)(2a - 3)$ **41.** $(2a + 3)(a + 6)$
42. $(2a + 3)(2a - 3)$ **43.** $(6b + c)(6b - c)$
44. $(5 + a)(5 - a)$ **45.** $4(2x + y)(2x - y)$
46. $(3x + y^2)(3x - y^2)$ **47.** $(x^2 + 4)(x + 2)(x - 2)$
48. $(y^2 + 9)(y + 3)(y - 3)$ **49.** $(b + 6)^2$ **50.** $(c - 5)^2$
51. $(2x - 3)^2$ **52.** $(3x - 2)^2$ **53.** $(R + 2S)(R^2 - 2RS + 4S^2)$
54. $2(2x - 3)(4x^2 + 6x + 9)$ **55.** $(3a + 5b)(9a^2 - 15ab + 25b^2)$
56. $(xy - 1)(x^2y^2 + xy + 1)$ **57.** $2(x^3 + 5)(x^6 - 5x^3 + 25)$
58. $(4x^4 - y^5)(16x^8 + 4x^4y^5 + y^{10})$
59. $(ab^2 + c^3)(a^2b^4 - ab^2c^3 + c^6)$ **60.** $3x^3(4x - 1)$
61. $(a - 5)(a + 2)$ **62.** $(4a - 1)(a - 5)$
63. $(3y + 2)(3y - 2)$ **64.** $(2a + 3b)(3x - 2)$
65. $(b - 5)(b + 4)$ **66.** $(3x + 2)(3x + 5)$ **67.** $(a + 7)^2$
68. $3x^3(2x + 1)(2x - 1)$ **69.** $c(c + 4)(c + 5)$ **70.** $(4a - 1)^2$
71. $(b^2 + 1)(b + 1)(b - 1)$ **72.** $\{1,-3\}$ **73.** $\{0,8\}$
74. $\left\{-\frac{1}{5},\frac{7}{3}\right\}$ **75.** $\left\{\frac{1}{7},\frac{8}{5}\right\}$ **76.** $\left\{\frac{4}{3},9\right\}$ **77.** $\{0,-9,-4\}$
78. $\left\{\frac{4}{5},-\frac{4}{5}\right\}$ **79.** $\{-1,3,-2\}$ **80.** $\left\{0,\frac{9}{4}\right\}$ **81.** $\{-1,1\}$
82. $\{0,64\}$ **83.** $\{-5,5\}$ **84.** $\{6,-5\}$ **85.** $\{1\}$ **86.** $\left\{-\frac{1}{4},-3\right\}$
87. $\left\{-\frac{2}{5},2\right\}$ **88.** $\{4\}$ **89.** $\{1,3\}$ **90.** $\left\{-1,\frac{3}{4}\right\}$ **91.** $\{2,-3\}$
92. 9,10 **93.** 8 feet and 13 feet **94.** 20 cattle **95.** 6 seconds

Chapter 4 cumulative test

1. 33 **2.** $8a^6b^3$ **3.** $a^2 + 4ab + 4b^2$ **4.** a^6 **5.** 12 **6.** x^5y^4
7. $5x - 7y$ **8.** $9x^2 - 4y^2$ **9.** $\frac{8a^6}{b^3}$ **10.** $\frac{1}{x^2}$ **11.** $4x + 4y$
12. $\frac{x^6}{9y^4}$ **13. a.** 0 **b.** 0 **c.** -6 **14.** $\left\{\frac{9}{5}\right\}$ **15.** $x < 12$
16. $\{7\}$ **17.** $x \geq 3$ **18.** $\{-3,3\}$ **19.** $\{2,5\}$ **20.** $x < \frac{5}{2}$
21. $2 \leq x \leq \frac{14}{3}$ **22.** $x = 3y$ **23.** $x = \frac{5y + 2}{3}$
24. $2ab(1 - 2ab - 4a^2b^4)$ **25.** $(2a + 3)^2$
26. $(5c + 3d)(5c - 3d)$ **27.** $(2a + 3)(2a - 5)$
28. $(x + 3)(x + 6)$ **29.** 14,39 **30.** 11,13
31. \$10,000 at 8%; \$5,000 at 6% **32.** 6 meters by 11 meters

Chapter 5

Exercise 5–1

Answers to odd-numbered problems

1. $\frac{1}{3}$ **3.** undefined **5.** 4 **7.** $\frac{13}{23}$ **9.** $\frac{16}{17}$ **11.** 3
13. undefined **15.** $\frac{15}{2}$ **17.** all real numbers except 0
19. all real numbers except 5 **21.** all real numbers except -3
23. all real numbers except $\frac{3}{4}$ **25.** all real numbers except $\frac{8}{3}$
27. all real numbers except 1 and 6 **29.** all real numbers except -2 and $\frac{4}{3}$ **31.** all real numbers except $-\frac{2}{3}$ and $\frac{2}{3}$
33. all real numbers except -3 and 3 **35.** all real numbers except $\frac{7}{3}$ **37.** all real numbers except -1 and 1 **39.** all real numbers **41.** $L = 0$ **43.** $T_1 \neq 0, T_2 \neq 0$

Solutions to trial exercise problems

6. $\frac{-5b^3}{5 - 2b}$; $b = -2$

$$\frac{-5b^3}{5 - 2b} = \frac{-5(-2)^3}{5 - 2(-2)} = \frac{-5(-8)}{5 + 4} = \frac{40}{9}$$

9. $\frac{(-2x)^2}{x^2 + 3x + 7}$; $x = 2$

$$\frac{(-2x)^2}{x^2 + 3x + 7} = \frac{[(-2)(2)]^2}{(2)^2 + 3(2) + 7} = \frac{(-4)^2}{4 + 6 + 7} = \frac{16}{17}$$

22. $\frac{x + 1}{2x - 1}$ Set $2x - 1 = 0$, then $2x = 1$ and $x = \frac{1}{2}$.

Domain is all real numbers except $\frac{1}{2}$.

25. $\frac{y + 4}{8 - 3y}$ Set $8 - 3y = 0$, then $3y = 8$ and $y = \frac{8}{3}$.

Domain is all real numbers except $\frac{8}{3}$.

28. $\frac{5s^2 + 7}{2s^2 - s - 3}$ Set $2s^2 - s - 3 = 0$ and factor. We have

$(2s - 3)(s + 1) = 0$. Then
$2s - 3 = 0$ or $s + 1 = 0$
$2s = 3$ $\quad s = -1$
$s = \frac{3}{2}$ $\quad s = -1$

Domain is all real numbers except -1 and $\frac{3}{2}$.

32. $\frac{a - 2}{4a^2 - 16}$ Set $4a^2 - 16 = 0$ and factor. We have
$4(a - 2)(a + 2) = 0$.
This is true if and only if $a - 2 = 0$, $a = 2$ or $a + 2 = 0$, $a = -2$.
Domain is all real numbers except -2 and 2.

37. $\frac{17q}{3q^2 - 3}$ Set $3q^2 - 3 = 0$ and factor to get
$3(q^2 - 1) = 0$
$3(q + 1)(q - 1) = 0$
$q + 1 = 0$ or $q - 1 = 0$
Then $q = -1$ or $q = 1$
Domain is all real numbers except -1 and 1.

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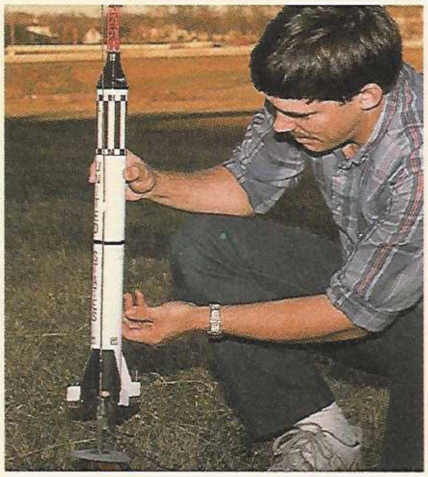

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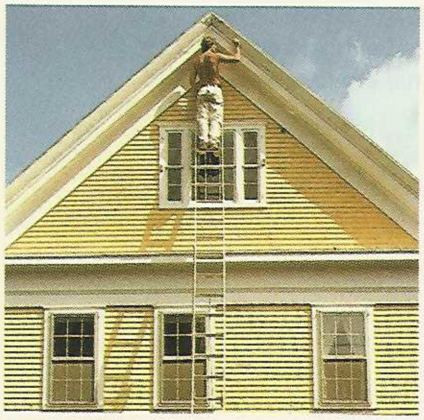

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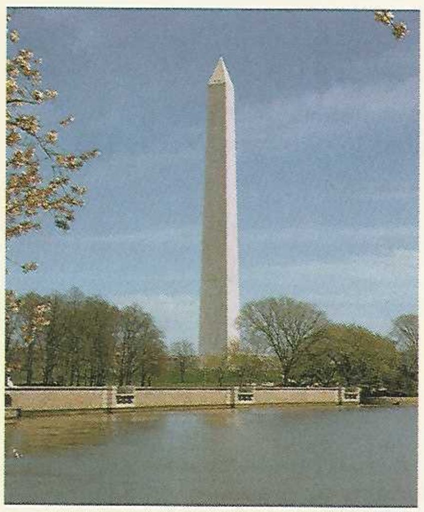

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